ARITHMETIC DATA FOR ANYONIC SYSTEMS

AND TOPOLOGICAL PHASES

by

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This dissertation is dedicated to
my wife Katherine and children
Taliesin, Gwenifryd, and Colin.
They have all been very good sports
about the whole thing.

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by

MATTHEW TITSWORTH, BS, MS

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Topological phases (TPs) are physical systems described by topological quantum field theories (TQFTs) and give rise to anyons. These anyons can then be used to perform topological quantum computation. All of these things can be studied concretely using the arithmetic data of a unitary fusion category $\mathcal{C}$ and its monoidal center, $\mathcal{Z}(\mathcal{C})$, which is a modular tensor category.

In this dissertation, we study this arithmetic data and the problem of determining when two sets correspond to the same fusion (modular) category. This process requires stating the question carefully in the language of geometric invariant theory, from which an answer immediately arises. We use these tools to develop a classification at the fusion level of anyonic systems which are of interest to topological quantum computing.

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CHAPTER 1

INTRODUCTION

Ostensibly, this dissertation is about topological phases of matter, a kind of quantum system restricted to (effectively) exist only in two dimensions and the low energy excitations that arise therein, called anyons. Effectively, it is about mathematical objects called fusion categories and their related structures. The beauty of this is it is essentially an exercise in linguistics to switch between them; for those properties of interest, one can write down dictionaries which faithfully translate between the two. This has faciliated a remarkable influence of each upon the other that has influenced the development of both.

As with many things, the proper choice of language can be key; ask a question in the right way and the answer becomes obvious. This choice will play an important role in the results presented. By employing the language of schemes used in chapter 4 the results of chapter 5 become immediate which makes 6 amazingly simple. In this way, this dissertation can be seen, in part, as an extended argument for the value of engaging with this formulation.

The algebro-geometric foundations extended and used herein are fundamental. They provide elementary descriptions of both anyons and fusion categories. However there has been a reluctance to engage with this "schemey" formalism because of the apparent complexity on the face of things. To our knowledge, our work in (Hagge and Titsworth, 2015) (contained in chapters 4 and 5) is the first time anyone has stepped back to consider these foundations from that perspective because in doing so the primary results become, almost embarrassingly, obvious. The details in getting there can be daunting, certainly the work went in unexpected directions, but when all is said and done there is a snickering elegance to the whole thing.

1.1 What is a Topological Phase?

Topological phases (TPs) are quantum systems with topological order (Wen, 1990). This means their effective field theory is a *topological quantum field theory* (TQFT, introduced in (Witten, 1989)). They cannot be characterized by broken symmetries and require the exploration of new properties for classification. These are ground state degeneracy, non-Abelian Berry's phase, and quasi-particle excitations – called *anyons* – with fractional statistics. Topological phases come in two basic flavors: intrinsic (Wen, 1990), in which the ground state degeneracy is protected by the topology of the system, and symmetry protected (Chen et al., 2011), if the ground state degeneracy is sensitive to symmetry breaking (see also (Liu et al., 2013)). The two can be unified through the notion of *symmetry enriched topological phases*(Barkeshli et al., 2014).

Examples of intrinsic topological phases are proposed to include frustrated quantum magnets (Moessner and Sondhi, 2001b,a; Levin and Wen, 2005), Fractional quantum hall liquids (Arovas et al., 1984; Halperin, 1984; Moore and Read, 1991; Wen, 1991) and certain high T_c superconductors (Wen, 1990). The primary examples of symmetry protected phases are topological insulators and superconductors (Kane and Mele, 2005; Fu et al., 2007; Schnyder et al., 2008; Kitaev et al., 2009).

1.1.1 Prologue to Topological Phases

The origin of the story goes back to (Dirac, 1931), in which Dirac shows that the existence of magnetic monopoles is consistent with quantum mechanics as formulated and that their existence would necessitate quantization of the electric charge. While there are more approachable treatments (see (Sakurai and Napolitano, 2011) for example) from a physical perspective, Dirac's is interesting because his construction makes direct use of the system's topology. It was very quickly realized (Saha, 1936, 1949) that electron-monopole composistes exhibited a peculiar angular momentum – namely that it was shifted by a factor of $\frac{1}{2}$.

1.1.2 The Emergence of Anyons

This was resolved by Wilczek in (Goldhaber, 1976; Wilczek, 1982a,b) by developing the generalized spin-statistics theorem of (2.1) and noting this as a consequence of the fact that $SO(2)$ – the group of rotations in two-dimension – has no non-trivial commutation relations. As such it does not restrict the possible values that can be induced by an exchange of particles. To be very short (see (Wilczek, 1990) for details), it was noticed that in two dimensions, the statistics of n identical particles were described by one dimensional representations of the braid group B_n rather than the symmetric group S_n.[1] These particles were called anyons. It was later shown (Fröhlich and Marchetti, 1987, 1988b,a, 1989) that higher dimensional representations of B_n were acceptable as descriptions of anyon statistics. Those with one dimensional representations are referred to as "Abelian" while those with greater dimension are referred to as "non-Abelian."

An important question was then how states giving rise to anyonic excitations could be realized. The fractional quantum hall effect(Tsui et al., 1982) was quickly proposed to support Abelian excitations (Arovas et al., 1984; Halperin, 1984). The idea that non-Abelian anyons could be physically realized in FQHE systems came from (Moore and Read, 1991; Wen, 1990). Anyonic excitations in chiral spin liquids and High T_c superconductors were proposed in (Wen et al., 1989).

Mathematically, there were a number of constructions which gave rise to anyons – σ-models with Hopf terms (Wilczek and Zee, 1983), gauge theories with Chern-Simons terms(Witten, 1989), and rational conformal field theories (Moore and Seiberg, 1989). Topological order was introduced by (Wen, 1990) and characterized the ground state of chiral spin liquids and FQHEs as being described by the TQFTs of (Witten, 1989). Physically,

[1] For $n \in \mathbb{N}$ the braid group B_n has generators σ_1, σ_{n-1} that obey the relations $\sigma_i\sigma_j = \sigma_j\sigma_i, |i-j| \geq 2$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, 1 \leq i \leq n-2$. The symmetric group S_n can be presented with the same generators and an additional relation that $\sigma_i\sigma_i = 1$.

this starts with systems whose ground state degeneracy is not characterized by an order parameter, i.e., those which do not have a complete description in terms of Ginzburg-Landau theory (Ginzburg and Landau, 1950; Landau, 1969). This ground state is robust against local perturbations and contains anyonic quasi-particles. Mathematically, there are numerous constructions (Wess and Zumino, 1971; Dijkgraaf and Witten, 1990; Reshetikhin and Turaev, 1991a; Turaev and Viro, 1992a; Barrett and Westbury, 1996) for TQFTs including ones which do not come from Chern-Simons theories(Hong et al., 2007). There is much overlap between them, see (Bakalov and Kirillov, 2001) for exposition.

1.1.3 Describing Topological Phases

Jumps in physical understanding were made when (Kitaev, 2003) and later (Levin and Wen, 2005) (the former shown to be a subclass of the latter in (Buerschaper and Aguado, 2009; Balsam and Kirillov, 2012)) developed a method for constructing models of TPs which realize the TQFTs of (Turaev and Viro, 1992a). These models, called *String-nets*(Levin and Wen, 2005), are a Hamiltonian method for realizing a large class of (2+1) dimensional topological phases called "doubled" or "non-chiral" topological phases. String-nets are lattice models labeled by objects in a *unitary fusion category* (UFC, see (Wang, 2010)) $\mathcal{C}$.[2] Quasi-particle excitations are given by a *unitary modular tensor category* (UMTC, see (Bakalov and Kirillov, 2001; Turaev, 2010; Wang, 2010)) called the *monoidal center* $\mathcal{Z}(\mathcal{C})$ (Majid, 1991; Joyal and Street, 1991; Müger, 2003c). In work by Walker and Wang (Walker and Wang, 2011), the construction of (Levin and Wen, 2005) was extended to the (3+1) dimensional case and defined on unitary ribbon categories (URC) – a class which subsumes UMTCs. Meanwhile, (Freedman et al., 2011) developed a way to construct these same topological phases "off lattice."

[2] A *Category* $\mathcal{C}$ is a mathematical object consisting of a class $\mathcal{C}_0$ of "objects" in the category and for every pair of objects a, b in the category a set $\mathcal{C}(a, b)$ of "morphisms", i.e., $f : a \to b$. For every $a \in \mathcal{C}_0$ there is an "identity on a" $1_a : a \to a$. For every triple a, b, c with an $f : a \to b$ and $g : b \to c$ there is a map $h \equiv g \circ f : a \to c$ such that $h \circ (g \circ f) = (h \circ g) \circ f$. See (MacLane, 1998) for background.

In work by Kitaev and Kong(Kitaev and Kong, 2011) the string-net models of (Levin and Wen, 2005) were expanded to include topological phases with boundary. This extension allows for the description of defects and domain walls (See also (Bais et al., 2009; Bais and Slingerland, 2009)). It is shown in (Kitaev and Kong, 2011) and (Kong, 2014) that excitations on a gapped boundary are described by a *module category* $\mathcal{M}$ over $\mathcal{C}$ (Ostrik, 2003). The author is unaware of an analog to (Kitaev and Kong, 2011) for (Walker and Wang, 2011) or (Freedman et al., 2011).

It was also discovered (Kitaev and Preskill, 2006; Levin and Wen, 2006; Chen et al., 2010) that the existence of topological order without symmetry (as described by (Wen, 1990), also called *intrinsic topological order*) could be characterized by the existence of *long range entanglement*. A system is said to be *long range entangled* if the Hilbert space can not be transformed to a direct product of vector spaces through a local unitary evolution.[3] It is argued in (Chen et al., 2010) that the orbits of these local unitary evolutions define equivalences among string-nets and correspond to their universality classes. A separate notion of equivalences among anyonic excitations in topological phases described by string-nets is given by (Kitaev and Kong, 2011) as the ability to construct transparent domain walls, i.e those through which anyons can pass without being transmuted.

1.1.4 Anyons and Quantum Computation

Aside from being of intellectual curiosity, anyons provide a promising route towards quantum computation(Nielsen and Chuang, 2000). Given an appropriate system of non-Abelian anyons, it was pointed out in (Kitaev, 2003) that they could be used to efficiently simulate a quantum circuit(Deutsch, 1989). (Freedman et al., 2002) showed that this works the other way as well: a quantum circuit can efficiently simulate a system of anyons. This then demonstrates that the two have equivalent computational power.

[3]Local unitary evolutions on a string-net can be defined by the operation (2.12) as well as a rule for resolving loops.

Anyons have an inherent advantage over other quantum computing schemes in that can store information non-locally and are thus fault-tolerant by design. Anyons have a property called fusion, which means that when you bring them sufficiently close together the composite state-vector decomposes as a super-position of a fixed set anyons which is invariant for the original pair. Leave them far enough apart, and the likelihood of this happening essentially goes to zero.

When using anyons for quantum computing, the information for the computation is stored in the possible decompositions resulting in the fusion of two anyons. The operations performed on that information are implemented by braiding of anyons around one another. The result of the computation is obtained by measurement of the resulting state. It is possible to perform these computations in such a way that only a single measurement needs to be made, and so the computational information is not otherwise available even though it is being operated on.

Finding a system of anyons capable of universal quantum computing has turned out to be a daunting task. In order for an anyonic system to be able to efficiently simulate a quantum circuit model it needs to be able to simulate arbitrary unitary operations to arbitrary accuracy. The basic condition for this to be possible seems to be that the dimension of the braid group representation obtained from the anyon statistics needs to be infinite(Naidu and Rowell, 2010). Most of the realistically feasible systems have finite dimensional image, however there are schemes(Cui et al., 2015; Cui and Wang, 2015) which would supplement braiding with measurement and return universality.

1.2 Outline

The structure of this dissertation is as follows. Chapters 2 and 3 are expository. Chapter 4 is a mixture. The material of 4.1 exists in the literature. The material of 4.2 is known, but to our knowledge we are the first ones to carefully write down the definitions in such

a way. Chapter 5 is taken from (Hagge and Titsworth, 2015), by the author and Tobias Hagge, submitted to Journal of Algebra. Chapter 6 is taken from joint work by the author, Eddy Ardonne, and Peter Finch, in final preparation. Chapter 7 looks at future directions for continued research.

CHAPTER 2

HOW TO BUILD AN ANYONIC SYSTEM

In this chapter we will focus on the properties of anyonic systems and on constructing string-net models which realize them. To help provide context and develop intuition along the way, we will focus on simple examples which will help to develop intuition behind the material developed in chapter 4. An illustrative example will be the $\mathbb{Z}_2$ lattice gauge theory, also known as the toric code. This example is the easiest to visualize as implementable in physical systems. It is also the only example for which all levels of analysis presented in this chapter have previously been completed in literature(Kitaev and Kong, 2011).

The downside of the toric code is that many properties of the toric code are not shared with even slightly more complicated models. Because of this, we will also also consider in this section examples related to the theories $D(\mathbf{Fib})$ and $D(S_3)$. The details for these are discussed in their own sections. Then, as necessary in subsequent chapters, we will expand these examples appropriately.

2.1 Anyons, Anyone?

The defining feature of anyons can be thought of as this: Let $|\psi_1, \psi_2\rangle$ be the state vector for a system consisting of two identical particles in statues ψ_1 and ψ_2 respectively. Under the normal spin-statistics theorem, we have that

$$|\psi_1, \psi_2\rangle = \pm|\psi_2, \psi_1\rangle$$

where the result is $+1$ when the particles are bosonic and -1 when the particles are fermionic.

In anyonic systems, this relationship changes. Instead, we have

$$|\psi_1, \psi_2\rangle = e^{\pm\imath\theta}|\psi_2, \psi_1\rangle \quad \text{or} \tag{2.1}$$

$$|\psi_1, \psi_2\rangle = \mathbf{U}|\psi_2, \psi_1\rangle \tag{2.2}$$

where θ is a rational multiple of π (and thus $e^{\imath\pi\theta}$ is a root of unity) and $\mathbf{U}$ is a unitary matrix. The choice of sign for θ depends upon the direction of the exchange.

To see this, let's take a top down view of what happens when the particles ψ_1 and ψ_2 exchange places:



(2.3)

When ψ_1 is moved around ψ_2 in three dimensions, what matters is just the final position of the two particles. This is because the path can always be deformed in to one where ψ_1 does not encircle ψ_2. However, as shown in (2.3), the situation is different in two dimensions. Since ψ_2 is in the way, the path can't be deformed and the direction matters.

The case in (2.1) is referred to as "Abelian" anyonic statistics. A primary example is the Laughlin state in the $\nu = \frac{1}{3}$ fractional quantum Hall Effect (FQHE) (Camino et al., 2005, 2007). The case in (2.2) is referred to as "non-Abelian" anyonic statistics, since successive exchanges of multiple anyons are not commutative. There is evidence for non-Abelian excitations in the $\nu = \frac{5}{2}$ FQHE (An et al., 2011; Willett et al., 2013).

2.2 Algebraic Properties of Anyons

A primary feature of anyonic systems is the fusion and splitting of quasi-particles(Moore and Read, 1991). Given a set of anyon types L, the collection of ways in which anyons can fuse define what's called a set of *fusion rules*. These rules, together with the other information detailed in this section will be called a "theory of anyons" and for completely transparent reasons we will refer to the theory as a whole by $\mathcal{C}$.

Let a and b be labels for two anyons, also called their *anyonic charge* and define the state space for the pair to be V_{ab}. When two anyons are brought "close" together they

"fuse" into a superposition of possible single anyon states. This is directly analogous to the Clebsch-Gordan decomposition of spin systems. As such, the space V_{ab} can be decomposed into blocks

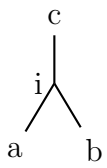
$$V_{ab} = \bigoplus_c V_{ab}^c \tag{2.4}$$

where c is a possible outcome of the fusion.

The dimension $dim(V_{ab}^c) =: N_{ab}^c$ is the number of fusion channels through which a and b can produce an anyon of type c. The collection of N_{ab}^c for all combinations of a, b, and c define what are called the *fusion rules*. This can be rewritten so that the rules for fusing an anyon of type a with an anyon of type b on the left are given by the matrix N_a with entries N_{ab}^c.

The fusion of anyons can also be described using pictures – commonly referred to as "string diagrams". These have their origin in the graphical calculus of (Penrose and Rindler, 1987) and were suitably generalized by (Joyal and Street, 1991). Since anyons are particles in two dimensions, these can be perfectly well thought of as world-lines for particles arranged in a 1D chain across the page and manipulated in the plane perpendicular to the page. A demonstrative picture can be found in Figure 2.1.

For each basis vector $|\psi_{ab}^{c;i}\rangle$ of each summand in (2.4) we draw a trivalent graph



$$\triangleq |\psi_{ab}^{c;i}\rangle \in V_{ab}^c \tag{2.5}$$

where i runs from 1 to N_{ab}^c and each picture represents a basis vector in the space V_{ab}^c. Thinking of these as world lines, these trivalent diagrams can be thought of as operators which collapse the pair a and b to the state c. A complete accounting of the relationship between this graphical notation and Dirac notation is given in (Kitaev, 2006). It is important to note here that unlike (Kitaev, 2006) and many others, we are not assuming that N_{ab}^c is either 0 or 1, i.e., we are not assuming that the fusion spaces are all one dimensional.

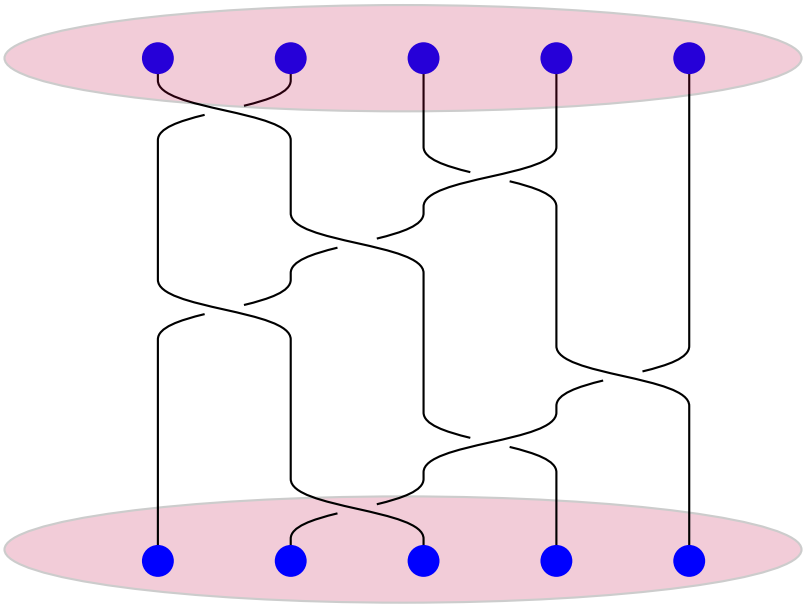
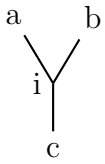


Figure 2.1. Braiding of anyons on a line.

It is also important to note the conventions being used. We are specifically describing fusion of anyons, and considering the diagrams as world lines they are to be read from bottom to top. Similarly, one creates anyons by splitting them apart. We can represent this diagrammatically by

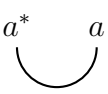


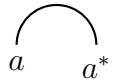
$$\triangleq |\psi^{ab}_{c;i}\rangle \in V^{ab}_{c}. \tag{2.6}$$

Physically, the system must have a ground-state in which there are no excitations – i.e., a vacuum sector, which we'll call **1**. It's also clear that fusion or splitting of an anyon with the vacuum sector has to be trivial. This requires that $V^{a}_{\mathbf{1}a} \cong V^{a}_{a\mathbf{1}} \cong \mathbb{C}$. Diagrammatically **1**, by convention, is either omitted or occasionally denoted by a dashed line.

For each anyon type, there are two special cases which need mentioning: those which capture the creation and annihilation of pairs of quasi-particles. The creation and annihilation diagrams, also called birth and death among other things, are then some fixed multiple of trivalent basis diagrams where the single incoming or outgoing strand is labeled by the

vacuum.

(birth) 

(death) 

(2.7)

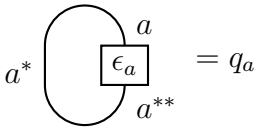
When there are more than two anyons, we can describe basis vectors for the composite space by assembling new pictures from old ones. The rules for this are that every vertex is trivalent, and the outgoing edge from a picture on the bottom must match an incoming edge for a picture on the top. Given that fusion of two anyons leads to a superposition of single particle states, superposition will also imply that the fusion of an arbitrary number of anyons can be decomposed as a superposition of single particle states. Again, this is directly analogous to Clebsch-Gordan decomposition.

As a first example, we have a pair of identities which, for suitable choice of births and deaths:



(2.8)

These two maps also let us define



$= q_a$ (2.9)

where, for the moment, the term ϵ[1] can be thought of as a correction factor for any difference in phase between the birth and death on a^*. It's possible to choose the creation and annihilation operators in such a way that all $\epsilon_a = \pm 1$, however, as we will see, there are theories of anyons in which they cannot all be made 1.

q_a is called the *quantum dimension* of a, and the collection of quantum dimensions satisfy the relation

$$q_a q_b = \sum_c N_{ab}^c q_c. \tag{2.10}$$

[1]These have their own consistency condition, but it is not terribly physically informative.

With this we also define the square of the quantum dimension of the whole theory to be

$$q(\mathcal{C})^2 = \sum_a (q_a)^2. \tag{2.11}$$

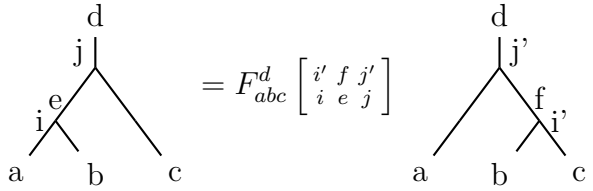
There's an oddity that arises though. Consider the case when fusing three types of anyons: a, b, and c. Since the fusion of two anyons leads to a superposition of single particle states, we must have that

$$V_{abc} = \bigoplus_d V_{abc}^d.$$

From the rules of our diagrams, given a set of basis vectors for the spaces V_{ab}^c, there are two possible ways to write the basis vectors for the space V_{abc}^d. In the first, a and b can create some particle e and the possible d are determined by those e such that V_{ab}^e and V_{ec}^d are not zero-dimensional. In the second, b and c can create some particle f and the possible d are determined by those f such that V_{bc}^f and V_{af}^d are not zero-dimensional. This then requires that

$$V_{ab}^e \otimes V_{ec}^d \cong V_{abc}^d \cong V_{cb}^f \otimes V_{af}^d \qquad \text{with} \qquad N_{abc}^d = \sum_e N_{ab}^e N_{ec}^d = \sum_f N_{bc}^f N_{af}^d.$$

Thus, the two different orderings of quasi-particle fusion with the same result simply specify different bases for the space V_{abc}^d. Given this, we can describe the change of basis between them as



(2.12)

where the left picture describes one way for combining three anyons and the right picture describes the other way. Physically the internal fusions are indistinguishable and the two processes are related via the F-matrices F_{abc}^d whose entries are given by $F_{abc}^d \begin{bmatrix} i' & f & j' \\ i & e & j \end{bmatrix}$, called *10j-symbols*. When $N_{ab}^c = 1$ for any a, b, c, the entries $F_{abc}^d \begin{bmatrix} 1 & f & 1 \\ 1 & e & 1 \end{bmatrix}$ of the F-matrices are called

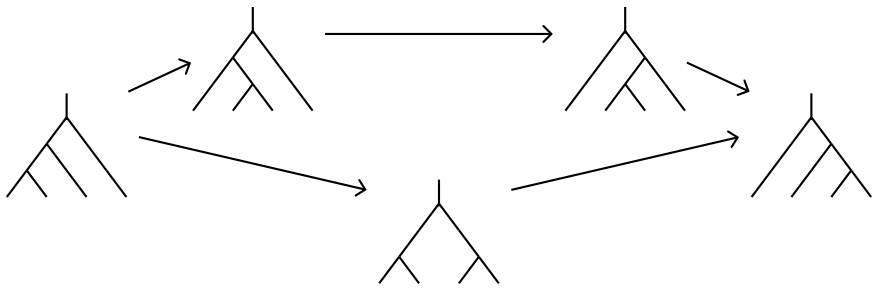


Figure 2.2. Basis changes between possible fusions of anyon types.

6j-symbols and also appear in the quantum gravity literature (See for example (Freidel and Louapre, 2003)). F by it self may be referred to as an F-symbol.

An additional complication arises when one tries to fuse four or more anyon types. As can be seen in Figure 2.2, there are two possible ways of changing bases from one in which all particles are fused from the left to one in which all particles are fused from the right. Since the two bases at the beginning and end of the process are identical, the two basis transformations have to be the same. The consistency condition this places on the F-symbols is sufficient to guarantee consistency for all other combinations.

The diagrammatic rules and conditions expressed above (with a similar condition to that in Figure 2.2 for splitting) allow us to construct the spaces

$$V_{a_1\ldots a_m}^{b_1\ldots b_n} \cong \bigoplus_c V_{a_1\ldots a_n}^{c} \otimes V_{c}^{b_1\ldots b_n}$$

for the fusion of anyons $a_1, \ldots, a_n$ in to anyons $b_1, \ldots, b_n$.

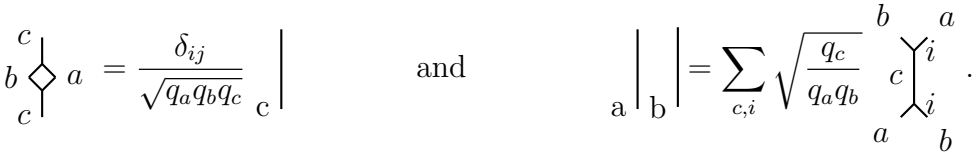
For physical systems, we require that the F matrices be unitary. This then implies that our fusion spaces are equipped with a Hermitian inner product, i.e., they are Hilbert spaces. If one chooses the normalization (see (Kitaev, 2006))

$$\langle \psi_{ab}^{c;j} | \psi_{ab}^{c;i} \rangle = \frac{1}{\sqrt{q_c q_a q_b}} \delta_{ij} \tag{2.13}$$

the result of this is that the inner product of $|\psi_{ab}\rangle = \sum_c \sum_i |\psi_{ab}^{c;i}\rangle$ in V_{ab} is $\langle \psi_{ab} | \psi_{ab} \rangle = 1$. The identity on V_{ab} is then given by

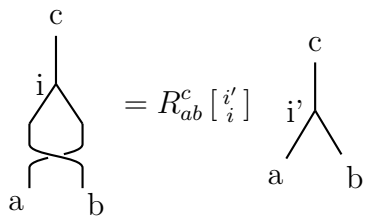
$$|\psi_{ab}\rangle\langle\psi_{ab}| = \sum_c \sum_i \left(\sqrt{\frac{q_c}{q_a q_b}} |\psi_{ab}^{c;i}\rangle\langle\psi_{ab}^{c;i}| \right). \tag{2.14}$$

Using our diagrammatic rules we can express these identities graphically as



This then makes it apparent that the splitting spaces V_c^{ab} are the dual spaces to the fusion spaces V_{ab}^c.

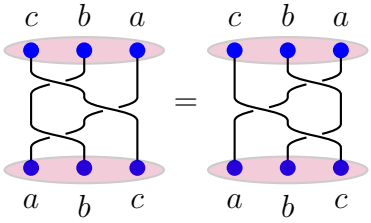
The exchange processes in (2.1) and (2.2) can also be described using pictures. In this case then, the relationship between $b \otimes a$ and $a \otimes b$ is given by



(2.15)

which serves as an ad-hoc definition for the $N_{ab}^c \times N_{ab}^c$ dimensional matrix R_{ab}^c. Since the act of moving a around b annihilates neither but simply evolves the state of the system, it's clear that this matrix must be unitary. Here, the choice of direction for the exchange is encoded by which "string" is on top.

As mentioned previously, the exchange of anyons yields a representation of the braid group B_n. As such, there are again consistency requirements. The matrix $R_{ab} \in V_{ab}^{ba}$ should be a direct sum of the matrices R_{ab}^c for every c such that $N_{ab}^c \neq 0$. These are then required to satisfy the Yang-Baxter equation:



as well as the condition that the order of disjoint exchanges is irrelevant. We can achieve this by making exchange operations compatible with fusion operations. Physically, this means

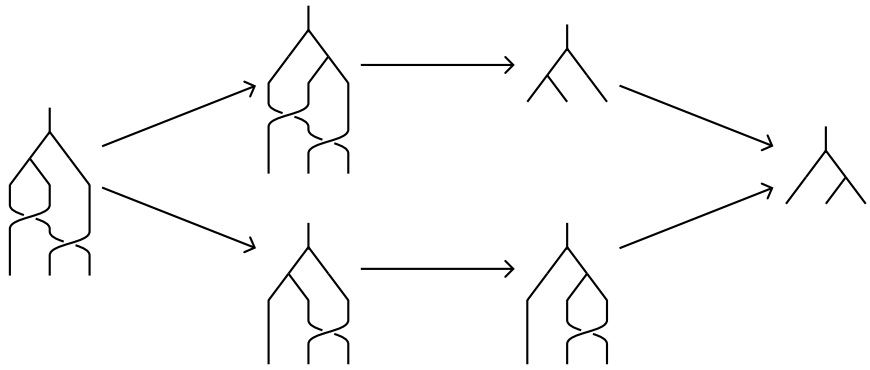


Figure 2.3. Possible resolutions of braidings. There is also a second diagram process in which all of the braidings are reversed.

that moving anyon d around the pair a, b which then fuse to c is the same as first fusing a and b to c and then moving d around c as in (2.16).



(2.16)

By fusing e and c at the top to get d, we then have diagrams in V_{abc}^{d} which gives us the constraint in Figure 2.3

We can combine all of these things to build other operations on quasi-particles. As an example, if we want to understand the exchange statistics for a particle of type a, we can look at the amplitude of the process in 2.4a. Specifically, whether or not a given quasi-particle obeys bosonic ($\theta_a = 1$), fermionic ($\theta_a = -1$), or some anyonic statistics is encoded in the θ information!

As another example, consider the operation in Figure 2.4b, where we create two pairs of particles, a, a^* and b, b^*, exchange a and b twice, then annihilate all of them. The amplitude for this process encodes the mutual statistics between different anyon types. It is actually through this process that we can detect anyons, provided that for particles of type a there's a particle of type b such that $R_{ab}R_{ba} \neq Id_{a\otimes b}$(Bonderson, 2007; Kitaev, 2006). This condition is satisfied by the requirement that the matrix S whose entries are the S_{ab} is itself unitary.



(a) Self statistics of anyon type a.

(b) Mutual statistics of anyon types a and b.

Figure 2.4. Self and mutual statistics for anyons.

It is conjectured(Bruillard et al., 2016) that the S_{ab}, θ_a and q_a contain all of the information necessary do distinguish a given anyonic system. By our rules above though, all of this information is contained in a consistent set of F's, R's, and ϵ's.

2.3 Examples

As previously mentioned, we will mostly focus on three examples: toric code, $D(\mathbf{Fib})$ and $Rep(D(S_3))$. Both the toric code and double Fibonacci theory can be built from two smaller theories – Z_2 and Fibonacci. In that case, we first present the Z_2 and Fibonacci theories and will build the toric code and Double Fibonacci as they arise.

The example $Rep(D(S_3))$ is more complicated. As the name suggests, it is related to the group S_3 however it is not built from S_3 in the same way. We will not go through the specifics for how to construct $D(S_3)$ from S_3, however they can be found in (Bakalov and Kirillov, 2001).

2.3.1 $\mathbb{Z}_2$ Theories

The data for $\mathbb{Z}_2$ theories is very simple. There are only two anyon types which we call 0 and 1 for obvious reasons. The fusion matrices are $N_{00}^0 = N_{01}^1 = N_{10}^1 = N_{11}^0 = 1$ with all other entries 0.

It is immediately apparent that all F and R matrices are 1-dimensional for Z_2 The are two sets of F-matrices for Z_2. Both have agree on all F matrices except F_{111}^1. These matrices

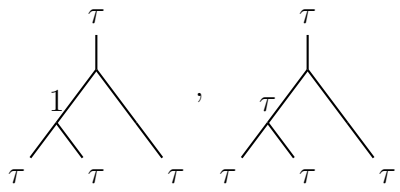
are all 1. In the Z_2 theory, F^1_{111} is also 1. The other theory – called the semion theory – has $F^1_{111} = -1$ We will primarily be concerned with the Z_2 theory. There are also two choices of R matrices for Z_2. These both also agree that $R^c_{ab} = 1$, except at R^0_{11}, which may be either 1 or -1. We will primarily be concerned with the Z_2 theory such that $R^0_{11} = 1$.

2.3.2 Fibonacci Theory

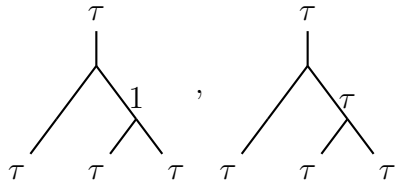
There are also only two anyon types for the Fibonacci theory. We will call them 1 and τ to be consistent with existing literature. The Fibonacci theory may also be referred to as $SU(2)_3/\mathbb{Z}_2$ or $SO(3)_3$.

The fusion matrices are $N^1_{11} = N^\tau_{1\tau} = N^\tau_{\tau 1} = N^1_{\tau\tau} = N^\tau_{\tau\tau} = 1$ with all other entries 0. In this case, not all F-matrices are one dimensional. We can see this by considering the fusion of three τ anyons.

Fusing the left first we obtain



and fusing the right first we obtain



We then have four 6j-symbols: $F^\tau_{\tau,\tau,\tau}\left[\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right]$, $F^\tau_{\tau,\tau,\tau}\left[\begin{smallmatrix}1&\tau\\1&1\end{smallmatrix}\right]$,$F^\tau_{\tau,\tau,\tau}\left[\begin{smallmatrix}1&1\\\tau&1\end{smallmatrix}\right]$, and $F^\tau_{\tau,\tau,\tau}\left[\begin{smallmatrix}1&\tau\\\tau&1\end{smallmatrix}\right]$. The only unitary F matrix for the Fibonacci theory is

$$F^\tau_{\tau,\tau,\tau} = \begin{pmatrix} \phi^{-1} & \sqrt{\phi^{-1}} \\ \sqrt{\phi^{-1}} & -\phi^{-1} \end{pmatrix}$$

with all other F-matrices 1. There is another solutions which does not admit unitary matrices, called the Yang-Lee theory, but we omit this.

There are also two choices for the R-matrices of the Fibonacci theory - $R^1_{\tau,\tau} = (-1)^{4/5}$, $R^\tau_{\tau,\tau} = -(-1)^{2/5}$ and its complex conjugate. Both share the same S matrix given by

$$\begin{pmatrix} 1 & \phi \\ \phi & -1 \end{pmatrix}$$

however they can be distinguished by the twist $\theta_\tau = e^{\pm 4\pi\imath/5}$. "The" Fibonacci theory, as in (Nayak et al., 2008) and others, refers to the $+4\pi\imath/5$ case.

2.3.3 S_3, $Rep(S_3)$ and $Rep(D(S_3))$

As previously mentioned, theories with fusion rules we'll call $Rep(D(S_3))$ are different, but important. First of all, they are the smallest anyon theories in for which all dimensions q_a are integers. Secondly, while they are not, by itself, universal for quantum computation they can be made so by incorporating measurement(Cui et al., 2015).

Another important difference is that unlike toric code and $D(\mathbf{Fib})$, fusion rules given by the group S_3 cannot be used to construct a theory of anyons. To see this, note that S_3 is not commutative, a requirement of the fusion rules for anyonic systems. However, we can write down a consistent set of F-symbols. Because each fusion only has one possible outcome, everything is one dimensional and so $F^d_{abc} = (1)$ is a valid solution. There are, additionally, five other solutions. These are classified by $\omega \in H^3(S_3), \mathbb{C}^\times)$, which is detailed in 4.1.1

Separately, and importantly, there is another set of fusion rules from which $Rep(D(S_3))$ can be built called $Rep(S_3)$. These have types $\{\mathbf{1}, \epsilon, \beta\}$, where $\{1, \epsilon\}$ forms a $\mathbb{Z}_2$ theory, and $\beta \otimes \beta = \mathbf{1} \oplus \epsilon \oplus \beta$. The fusion rules for $Rep(S_3)$ also do not yield any completely valid theories of anyons, however, there are different partial theories that can be distinguished by their arithmetic data.

At the fusion level there are three cases. In all three cases, a valid solution is

$$F^{\beta}_{\epsilon\beta\beta} = F^{\mathbf{1}}_{\beta\epsilon\beta} = F^{\beta}_{\beta\epsilon\beta} = F^{\mathbf{1}}_{\beta\beta\epsilon} = F^{\epsilon}_{\beta\beta\epsilon} = F^{\beta}_{\beta\beta\epsilon} = (-1)$$

$$F^{d}_{abc} = (1) \text{ otherwise.}$$

The remaining information which distinguishes them is

$$F^{\mathbf{1}}_{\beta\beta\beta} = \left(\omega^i\right), F^{\epsilon}_{\beta\beta\beta} = -\left(\omega^i\right), F^{\beta}_{\beta\beta\beta} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 2 \\ -1 & -1 & 2 \\ \omega^i & -\omega^i & 0 \end{pmatrix}$$

where $\omega^3 = 1$.

Only the case $\omega = 1$ admits a braiding, of which there are three. All can be distinguished by the twists $\{1, 1, \omega^i\}$. There are two possible S matrices

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix} \text{and} \qquad \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & -2 \end{pmatrix}$$

which correspond to the $\omega = 1$ and $\omega \neq 1$ cases respectively.

CHAPTER 3

STRING-NETS

There are many ways in the literature (see (Nayak et al., 2008) for a review and references therein) to construct specific systems with anyonic excitations. In this chapter, we will focus on the (2+1)D string-net models of Levin and Wen(Levin and Wen, 2005). Extensions by (Kitaev and Kong, 2011) to systems with boundaries and domain walls and (Walker and Wang, 2011) to (3+1)D systems will be omitted for clarity and concision. The motivated reader is encouraged to look there for some fascinating work.

In section 3.1, we define what a string-net is and the Levin-Wen Hamiltonian. In section 3.2, we define the anyons that arise in string-nets. Finally, in section 3.4, we define string-net models that realize the toric code and double Fibonacci theory.

3.1 What's a String-Net?

A string-net is just a spin model for constructing topological phases which realize a certain class of anyonic systems. At the ground level, they are built from part of the data detailed in the previous chapter, specifically a set of labels (types), a set of fusion rules, and valid F-moves. To that end we will reuse the terminology of that chapter while working on string-nets.

Definition 3.1.1. Let Σ be a closed surface with triangulation Δ. A *String-net* is a directed labeling on the vertices and edges of the planar graph dual to Δ (This is a trivalent graph we will call Λ) by objects of L as given above. A *String-net model* is the Hilbert space $V(\Sigma)$ whose basis vectors are string-nets together with the Hamiltonian

$$\mathcal{H} = \sum_{\text{vertices } i} (1 - Q_i) + \sum_{\text{plaquette } p} (1 - B_p). \tag{3.1}$$

Q_i and B_p are defined as follows:

1. Since Λ is a trivalent graph, consider the label at the vertex i and three edges surrounding it. These will form a trivalent graph like the one in (2.5). It is then natural to ask the question of whether or not this is a graph defined in (2.5) – called "admissible" and that's what Q_i does. Specifically, given a string-net $|\psi\rangle$, Q_i is the vertex operator such that

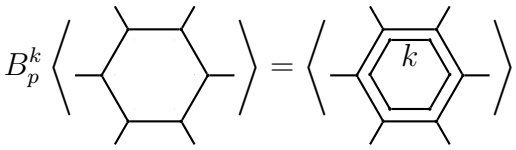
$$Q_i|\psi\rangle = \begin{cases} |\psi\rangle & \text{if the graph at vertex } i \text{ is defined in (2.5)} \\ 0 & \text{otherwise.} \end{cases}$$

   As per (Levin and Wen, 2005), Q_i can be thought of as an "electric charge" operator which prefers zero charge at the vertices. This constraint implies that the low energy Hilbert space is spanned by the set all string-nets constructed from graphs in (2.5).

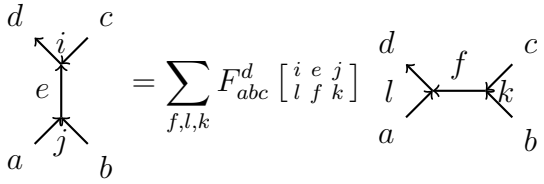
2. The operator B_p can be considered as a "magnetic flux" operator on the plaquette p. We define B_p as

$$B_p = \sum_{k \in L} \frac{q_k}{q(\mathcal{C})^2} B_p^k. \tag{3.2}$$

   The B_p^k operators can be described through their action on the plaquette p as inserting a loop of type k, i.e.,



   where the labels have been omitted for clarity (See (Levin and Wen, 2005) or chapter 11 of (Wen, 2007) for full details and eigenvalues of B_p^k). This diagram is then evaluated using $I - H$ moves as in



(3.3)

until a new linear combination of string-nets (without the loop) has been obtained. B_p^k evaluates to zero if the labelings on p are not admissible.

We note that the diagrams in (3.3) resemble those in (2.12), however these are directed. This is because, while our diagrammatic rules are basically the same, we are not working with world lines and have no preferred direction. Thus it is necessary to specify which are the incoming and outgoing edges.

The Q_i are Hermitian and projectors by definition and for a set of labels a, b, c they have N_{ab}^c eigenvalues. Since the B_p^k operators are evaluated by $I - H$ moves, their eigenvalues are sums of products of $10j$-symbols given in (2.5). They are Hermitian if all of the F-matrices are unitary. With the choice of coefficients in B_p^k in (3.2), the B_p are also projectors and the ground state satisfies $Q_i = B_p = \mathbf{1}$ for all vertices and plaquettes. Moreover, the Q_i and B_p all commute, and so commute with $\mathcal{H}$, and thus the Hamiltonian is exactly solvable.

The Hilbert space $\mathcal{L} \subset V(\Sigma)$ describing the ground state is then the space spanned by string-nets with admissible labelings. A string-net model is then fully determined by the collection of data (L, N, F, d, Δ).

3.2 Anyons in String-Net Models

The ground state of a string-net model satisfies $Q_i = B_p = 1$ for all vertices and plaquettes. Anyonic excitations then correspond to violations of these constraints for some *local* collection of vertices and plaquettes(Levin and Wen, 2005).

It was noted in (Levin and Wen, 2003) that the creation operator for a pair of anyons has an open string-like structure, with one end as the quasi-particle a and the other its anti-particle a^*. The location of this string is unobservable, and the energy penalty for creating such a pair depends only upon the endpoints. Graphically, an open string can be thought of as directed line segment labeled at one end by a and the other by a^*. The string is placed along the edges of the lattice, and then evaluated in the same manner as the B_p^s operators.

Unlike the B_p^s operators, general strings can follow an arbitrary path along the lattice. This means that we need to be able to lay them across vertices but that they only change the labeling on edges and vertices where they're lain. We also need that the side of the edge that their lain on, i.e., the vertices that they cross, doesn't affect the final labeling. This corresponds to the following picture:



The types of anyons supported by the string-net model (L, N, F, d, Δ) are those with this property, called "half-braiding." Half braiding can be defined by either of the pictures (but not necessarily both) in (2.16).

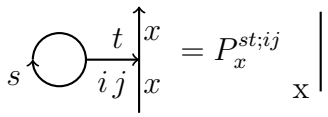
The relations necessary for the existence of a half braiding are of the form given in Figure 2.3 – specifically, that only one of the two possible (since we only move on top) processes must work. These anyons will define the same set of data as given in 2.1.

If the two end points of an open string coincide, then they form a closed string. Since the physical properties of the string-operator should depend only upon the end points, and a closed string has no endpoints, we should then require that they share an eigen-basis with the Hamiltonian, at least in the admissible sub space. The mutual commutation of the B_p^s operators then guarantees that the eigenstates of $\mathcal{H}$ in the $\mathcal{L}$ subspace are invariant under the addition of closed strings corresponding to objects with half-braidings.

3.3 Equivalence of String-Nets

There are several ways in which two string nets define the same topological phase. Consider two string-nets (L, N, F, d, Δ) and (L, N, F, d, Δ') defined on a surface Σ. They describe the same topological phase if there is a systematic way to take Δ to Δ'. The ways in which Δ can be changed are given by the $I - H$ move (3.3) and a rule for resolving "tadpoles" given

in



$$= P_x^{st;ij} \tag{3.4}$$

whose coefficients can be written in terms of F and d. The composition of these defines a local unitary transformation and in (Chen et al., 2010) it was shown that two string-net models (L, N, F, q, Δ) and (L, N, F, q, Δ') describe the same topological phase if they are connected by a local unitary transformation.

However, this is not the only way in which two string-nets may describe the same topological phase. As is shown in (Kitaev and Kong, 2011), it's possible that two string-net models (L, N, F, d, Δ) and (L', N', F', d', Δ) give rise to the same excitations, i.e., $\mathcal{Z}(L, N, F) \cong \mathcal{Z}(L', N', F')$. In this case the two string-net models are referred to as being "Morita equivalent." The full meaning of Morita equivalent will be given in 4.1.4.

3.4 Examples of String-Nets

Since the construction of open string-operators is not dependent upon the manifold Σ or its triangulation Δ, the anyonic excitations realized don't depend upon Σ or Δ either. This is not to say that Σ and Δ are not important. As seen above, they are explicitly part of the recipe for constructing the Hamiltonian. Additionally, the topological properties of Σ determine whether or not there are additional operations we can perform, e.g., moving an anyon around the hole in a torus. However, because the anyonic excitations are not dependent, we will omit them in discussing our examples below.

3.4.1 The Toric Code

The simplest example of a string-net is $\mathbb{Z}_2$ lattice gauge theory which can be defined from our $\mathbb{Z}_2$ theory in 2.3.1. This example is also known as the Toric code. It can be realized in the formalism of a spin $\frac{1}{2}$ system in the following way:

Consider a trivalent lattice[1] with spin $\frac{1}{2}$ particles decorating the edges. If there are N edges on the plaquette, $V(\Sigma)$ must then have dimension 2^n. We then define the Hamiltonian

$$\mathcal{H}_{tc} = \sum_v \prod_{i \in i(v)} \sigma_j^z + \sum_p \prod_{k \in \partial p} \sigma_k^x \tag{3.5}$$

where $i(v)$ is the set of all edges incident to vertex v, and ∂p is the set of all edges that define the boundary of plaquette p. The vertex term, given by $\sum_v (1 - \prod_{i \in i(v)} \sigma_j^z)$ introduces an energy penalty for states in which there are an odd number of $-\frac{1}{2}$ spins adjacent to a given vertex. This serves the purpose of dividing $V(\Sigma)$ from $\mathcal{L}$.

The plaquette term, $\sum_p \prod_{k \in \partial p} \sigma_k^x$, flips the spins on all edges around the plaquette. *States with the same energy will then be a sum of states on which the action of a plaquette operator is trivial.*

From this we can immediately make a connection to the string net model defined from the $\mathbb{Z}_2$ theory. The vertex term for this string net Hamiltonian identifies the spin $\frac{1}{2}$ with the label 0, and $-\frac{1}{2}$ with the label l. The plaquette term is subtly different, since in the string net model, we must consider B_p^0 and B_p^1. However, from the fusion rules, we see that fusion with the object 0 acts as the identity on the plaquette. The eigenvalue for this will always be 1 so long as the plaquette is admissibly labeled. Thus the space $\mathcal{L}$ for the string-net model corresponds to a sub-space for H_{tc}. Specifically, they share the same ground state.

The open and closed strings in the two models can also be compared. In both cases there are four types of open string operators, where one is trivial. For the Toric code, the first type of operator which we will call W_{tcv} create vertex defects at the end points of the strings and are defined via

$$\mathcal{W}_{tcv}(p) = \prod_{i \in p} \sigma_i^x \tag{3.6}$$

[1]For the toric code, the valence of the lattice is irrelevant. However we assume this here to maintain consistency with our definition.

which flips the spins along the path p. The second type of operator which we call W_{tcp}creates plaquette defects and is defined via

$$\mathcal{W}_{tcv}(p) = \prod_{ip} \sigma_i^z \tag{3.7}$$

where i is a plaquette corresponding to a path in the *dual* lattice. The final non trivial string operator creates a pair of vortex and plaquette defects. It is obtained by taking the sum of the two previous open strings, thus creating four types of excitation.

As mentioned in the last section, open strings in the string-net model correspond to linear combinations of operators labeled by elements of L which admit half braidings. For the Z_2 case, it is known that the N-matrices for these objects are given by the permutation representation for the Klein 4- $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ group. Thus, there are again four such operators. Since all $F^l_{ijk} = 1$, it's easy to see that there are three possible labels for operators: $0, 1, 0+1$, and that the solutions for the half braiding of 0 is 1, $0+1$ is 1 and 1 allows solutions of ± 1. Thus we again have four operators.

The theory of anyons we have just introduced is called the toric code. It is a theory with four anyon types and fusion rules given by the group $\mathbb{Z}_2 \times Z_2$. As with $\mathbb{Z}_2$ since there is only one possible outcome of each type of fusion, all F-matrices are one dimensional. There are several sets of F-matrices for the toric code, however the one related to $\mathbb{Z}_2$ are the ones with all matrices as 1.

3.4.2 Double Fibonacci

On the opposite end of the spectrum is the $SU(2)_3/\mathbb{Z}_2 \times SU(2)_3/\mathbb{Z}_2$ theory, also known as $D(\mathbf{Fib})$ for doubled Fibonacci. This is (predictably) built from the Fibonacci theory. The reason for this term "doubled" is explained in 4.1.4.[2]

[2]In fact, all of our examples are "doubled" phases.

While the physical Hilbert space of string-nets for $D(\mathbf{Fib})$ has the same dimension as the toric code – in both cases there are only two ways to label a given edge – the admissible subspace is very different. An immediate observation is that it must be of higher dimension than for the toric code, since it is allowable in the Fibonacci theory for three τ labels to meet at a point. However, since $N_{11}^{\tau} = N_{\tau 1} = N_{1\tau}^{1} = 0$, it is not admissible for a vertex to have only a single τ label. This assures that all pictures in the admissible subspace contain closed loops, however they may contain multiple closed loops that coincide along an edge.

We now look at excitations in $D(\mathbf{Fib})$. Excitations correspond to object-half braiding pairs. In the case of **Fib**, each object is braided, but there are two compatible braidings for it. Thus we have four object-half braiding pairs. The fusion rules for $D(\mathbf{Fib})$ are then given by the direct product of those for Fibonacci, which is to say that all four objects can be considered as pairs (a, b) where a and b are both objects of **Fib**. We identify these objects by $1 \equiv (1,1)$, $\tau \equiv (\tau, 1)$, $\bar{\tau} \equiv (1, \tau)$, $\alpha \equiv (\tau, \tau)$. The non-trivial fusion rules are then explicitly given by $N_{\tau\tau}^{1} = N_{\tau\tau}^{\tau} = N_{\tau\hat{\tau}}^{\alpha} = N_{\tau\alpha}^{\hat{\tau}} = N_{\tau\alpha}^{\alpha} = N_{\hat{\tau}\hat{\tau}}^{1} = N_{\hat{\tau}\hat{\tau}}^{\hat{\tau}} = N_{\hat{\tau}\alpha}^{\tau} = N_{\hat{\tau}\alpha}^{\alpha} = N_{\alpha\alpha}^{1} = N_{\alpha\alpha}^{\tau} = N_{\alpha\alpha}^{\hat{\tau}} = N_{\alpha\alpha}^{\alpha} = 1$. The F- and R- matrices for $D(\mathbf{Fib})$ can be completely computed from the F- and R-matrices for **Fib**. As a theory of anyons $D(\mathbf{Fib})$ is distinguished by the S and T matrices

$$S = \begin{pmatrix} 1 & \phi & \phi & \phi^2 \\ \phi & -1 & \phi^2 & -\phi \\ \phi & \phi^2 & -1 & -\phi \\ \phi^2 & -\phi & -\phi & 1 \end{pmatrix} \quad \text{and } T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (-1)^{4/5} & 0 & 0 \\ 0 & 0 & (-1)^{4/5} & 0 \\ 0 & 0 & 0 & -(-1)^{3/5} \end{pmatrix}.$$

3.4.3 *Rep*$(D(S_3))$

One can generalize the toric code approach of 3.4.1 by replacing the group $\mathbb{Z}_2$ with any other group G. The details of how this is done are given in (Kitaev, 2003, Section 4), and we refer the reader there for details. In this case the dimensionality of the Hilbert space associated

to each edge is $|G|$. As with the toric code, it is known that the excitations realized by this model are the same as those realized by the string-net model given by G or $Rep(G)$[3], formally proven in (Balsam and Kirillov, 2012, Theorem 4.1). This is an important observation as the Hilbert spaces in these cases have different dimensions.

Particularly, we are interested using S_3, the smallest non-Abelian group. We fix Σ and Δ and for simplicity, consider only the string-net models built from S_3 and $Rep(S_3)$. In the case of S_3, as with the Kitaev model, a 6-dimensional Hilbert space is associated to each edge in Δ. In contrast, the $Rep(S_3)$ model associates only a 3-dimensional Hilbert space to each edge.

While the lattice operators Q_i in both cases operate in essentially the same way, there is an important distinction in the B_p^k operators. It's easy to see that in the S_3 case all B_p^k operators are permutations – fusing the loop to the cell only has one possible outcome at each edge and all F-matrices are (1). In contrast, a B_p^β operator[4] on an n-sided face may have as many 3^n terms in its image.

The quasi-particle excitations for string-nets labeled by $Rep(S_3)$ and S_3 are as follows. For clarity, we consider $Rep(S_3)$. (Coste et al., 2000) present a construction for computing the data using S_3.

As previously mentioned, $\mathbf{1}$ and ϵ define a $\mathbb{Z}_2$ theory, specifically the case that $R^{\mathbf{1}}_{\epsilon\epsilon} = 1$. Additionally, the object $\mathbf{1} \oplus \epsilon$ yields an independent half-braiding, which will be called β_1. All three possible choices of ω^i in the braiding for give valid half-braidings, which will be referred to as β_2, β_3, and β_4. Finally, the objects $\mathbf{1} \oplus \beta$ and $\epsilon \oplus \beta$ yield independent half-braidings. These will be referred to as α_+ and α_-.

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[3]In this case, the operators for the Kitaev model are not as straightforward as in the $\mathbb{Z}_2$ case and so while both Hamiltonians have plaquette and vertex terms, the translation is not as exact.

[4]Recall that the objects of $Rep(S_3)$ are indexed by $\mathbf{1}$, ϵ, and β

The fusion rules are commutative and each β_i, together with $\{\mathbf{1}, \epsilon\}$, forms a copy of $Rep(S_3)$. We also get that $\beta_i \otimes \beta_j = \beta_k \oplus \beta_l$ with $i \neq j \neq k \neq l$. The remaining fusion rules are then given by

$$\epsilon \otimes \alpha_\pm = \alpha_\mp, \qquad \beta_i \otimes \alpha_\pm = \alpha_+ \oplus \alpha_-,$$
$$\alpha_\pm \otimes \alpha_\pm = 1 \bigoplus_{i=1}^{4} \beta_i, \qquad \alpha_\pm \otimes \alpha_\mp = \epsilon \bigoplus_{i=1}^{4} \beta_i$$

The automorphism group for $Rep(D(S_3))$ is $\mathbb{Z}_2 \times S_4$.

Predictably, there are a large number of F- and R- matrices for the $Rep(D(S_3))$ theory. However, since these have not appeared or classified prior to our work they are includes in Appendix A. The S- and T- matrices, however, are given by

$$S = \begin{pmatrix} 1 & 1 & 2 & 3 & 3 & 2 & 2 & 2 \\ 1 & 1 & 2 & -3 & -3 & 2 & 2 & 2 \\ 2 & 2 & 4 & 0 & 0 & -2 & -2 & -2 \\ 3 & -3 & 0 & 3 & -3 & 0 & 0 & 0 \\ 3 & -3 & 0 & -3 & 3 & 0 & 0 & 0 \\ 2 & 2 & -2 & 0 & 0 & 4 & -2 & -2 \\ 2 & 2 & -2 & 0 & 0 & -2 & -2 & 4 \\ 2 & 2 & -2 & 0 & 0 & -2 & 4 & -2 \end{pmatrix} \quad \text{and } T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 \end{pmatrix}. \tag{3.8}$$

Additionally, one may wish to consider labeling the string-net by any of the other possible solutions – any of the five other S_3 solutions or the two other $Rep(S_3)$ solution – any of the five other S_3 solutions or the two other $Rep(S_3)$ solutions. These also lead to valid theories of anyons, and can be related back to the Kitaev quantum double models via work in (Hu et al., 2013). In this case, one obtains the so called *twisted* double model with the same

fusion rules and modular data given by

$$S = \begin{pmatrix} 1 & 1 & 2 & 3 & 3 & 2 & 2 & 2 \\ 1 & 1 & 2 & -3 & -3 & 2 & 2 & 2 \\ 2 & 2 & 4 & 0 & 0 & -2 & -2 & -2 \\ 3 & -3 & 0 & -3 & 3 & 0 & 0 & 0 \\ 3 & -3 & 0 & 3 & -3 & 0 & 0 & 0 \\ 2 & 2 & -2 & 0 & 0 & 4 & -2 & -2 \\ 2 & 2 & -2 & 0 & 0 & -2 & -2 & 4 \\ 2 & 2 & -2 & 0 & 0 & -2 & 4 & -2 \end{pmatrix} \quad \text{and } T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \imath & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\imath & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega^2 \end{pmatrix}. \tag{3.9}$$

Remark 3.4.1. It was shown in (Gelaki et al., 2009) that there are two additional theories of anyons with these fusion rules, but these arise in a much different way which we exclude. Since the total set of solutions have not previously been classified before though, we include them with the others in the Appendix.

CHAPTER 4

FUSION CATEGORIES – A MATHEMATICAL TOOL FOR CONSTRUCTING MODELS OF TOPOLOGICAL PHASES

In this chapter we formalize many of the notions that have been mentioned in the previous chapters. This allows us to be precise about the exact conditions the arithmetic data for anyonic systems and string-nets must satisfy.

4.1 Fusion and Modular Categories

In the sequel, we assume k is an algebraically closed field of characteristic 0. Physically, we usually take $k = \mathbb{C}$, however it turns out that we will never need to use all of $\mathbb{C}$.

We first recall the definitions of based rings and fusion categories.

Definition 4.1.1 ((Ostrik, 2003, Definitions 2.1 and 2.2))**.** A *unital based ring* (R, B) is a $\mathbb{Z}^+$- ring R together with a set $B \subset R$ and identity $1_R \in B$ such that

- (Structure constants) There exist non-negative integers N_{XY}^Z for $X, Y, Z \in B$ such that

$$XY = \sum_{Z \in B} N_{XY}^Z.$$

- (Duality) A bijection $* : B \to B$ such that $1_R^* = 1_R$ which extends to an anti-involution on (R, B), i.e., $(XY)^* = X^*Y^*, \forall X, Y \in B$.

By associativity the structure constants must satisfy

$$\sum_{U \in B} N_{XY}^U N_{UZ}^W = \sum_{V \in B} N_{YZ}^V N_{XV}^W \tag{4.1}$$

for all $U, V, W, X, Y, Z \in B$.

The structure constants define a map $N : B^{\times 3} \to \mathbb{Z}^+$ which we can extend recursively to arbitrary $n+1 > 3$ via

$$N^Y_{X_1 \ldots X_n} := \sum_{Z \in B} N^Z_{X_1 \ldots X_{n-1}} N^Y_{Z X_n}$$

for all $X_1, \ldots, X_n, Y \in B$. Those structure constants which are non-zero will play an important role and so we define

$$\Gamma(R, B) = \{(X, Y, Z) \in B^{\times 3} | N^Z_{XY} \neq 0\}$$

with γ^Z_{XY} the notation for (X, Y, Z). We will say that (R, B) is *multiplicity free* if $N(\Gamma(R, B)) = \{1\}$.

For based rings (R, B) and (R, B'), every bijection $\rho : B \to B'$ satisfying

$$N^Z_{XY} = N^{\phi(Z)}_{\phi(X)\phi(Y)}, \forall X, Y, Z \in B \tag{4.2}$$

defines a unique based ring isomorphism $\rho' : (R, B) \to (R, B')$ and vice versa. Let $Aut(R, B)$ to be the group of based ring automorphisms of (R, B).

Definition 4.1.2. A monoidal category $\mathcal{C}$ is a category with a bifunctor $\otimes : \mathcal{C} \times C \to C$, an object $\mathbf{1}$, and natural isomorphisms $\alpha_{a,b,c}, \lambda_a, \rho_a$ satisfying

$$\begin{array}{ccccc}
a \otimes (b \otimes (c \otimes d)) & \xrightarrow[\alpha_{a,b,(c\otimes d)}]{} & (a \otimes b) \otimes (c \otimes d) & \xrightarrow[\alpha_{(a\otimes b),c,d}]{} & ((a \otimes b) \otimes c) \otimes d \\
\Big\downarrow {\scriptstyle Id_a \otimes \alpha_{b,c,d}} & & & & \Big\uparrow {\scriptstyle \alpha_{a,b,c} \otimes Id_d} \\
a \otimes ((b \otimes c) \otimes d) & & \xrightarrow{\alpha_{a,(b\otimes c),d}} & & (a \otimes (b \otimes c)) \otimes d
\end{array} \tag{4.3}$$

$$\begin{array}{ccc}
a \otimes (\mathbf{1} \otimes c) & \xrightarrow{\alpha_{a,\mathbf{1},c}} & (a \otimes \mathbf{1}) \otimes c \\
& {\scriptstyle Id_a \otimes \lambda_c} \searrow \quad \swarrow {\scriptstyle \rho_a \otimes Id_c} & \\
& a \otimes c &
\end{array} \tag{4.4}$$

$\mathcal{C}$ is semi-simple if $a \otimes b \cong \bigoplus_c c$ where c is an equivalence class of simple objects in $\mathcal{C}$. $\mathcal{C}$ is k-linear if $\mathcal{C}(a,b)$ is a (finite dimensional) vector space over k. $\mathcal{C}$ is rigid if for every equivalence class of simple objects there is a right dual a^* and a pair of morphisms $ev_a : a \otimes a^* \to \mathbb{C}$ and $coev_a : a^* \otimes a \to \mathbb{C}$ satisfying the constraints

$$Id_a = \lambda_a \circ (ev_a \otimes Id_a) \circ \alpha_{a,a^*,a} \circ (Id_a \otimes coev_a) \circ (\rho_a)^{-1} \tag{4.5}$$

$$Id_{a^*} = \rho_a \circ (Id_{a^*} \otimes ev_a) \circ \alpha^{-1}_{a^*,a,a^*} \circ (coev_a \otimes Id_{a^*}) \circ (\lambda_a)^{-1} \tag{4.6}$$

and similarly a left dual ${}^*a$ with maps $ev_a^{'}$ and $coev_a^{'}$ satisfying similar conditions.[1]

Given two monoidal categories $\mathcal{C}$ and $\mathcal{D}$, a central question is one of whether or not they are equivalent in some suitable sense. The strongest of these is *monoidal equivalence*.

Definition 4.1.3. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. A *monoidal functor* $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a triple $(\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2)$ where $\mathcal{F}_0$ is a functor from $\mathcal{C}$ to $\mathcal{D}$, $\mathcal{F}_1$ is an isomorphism between $\mathbf{1}$ and $\mathcal{F}_0(\mathbf{1})$, and

$$\mathcal{F}_2(a,b) : F(a) \otimes F(b) \to F(a \otimes b)$$

is a family of natural isomorphisms satisfying

$$\begin{array}{ccc}
\mathcal{F}_0(a) \otimes (\mathcal{F}_0(b) \otimes \mathcal{F}_0(c)) & \xrightarrow{\alpha_{\mathcal{F}_0(a),\mathcal{F}_0(b),\mathcal{F}_0(c)}} & (\mathcal{F}_0(a) \otimes \mathcal{F}_0(b)) \otimes \mathcal{F}_0(c) \\
\downarrow{\scriptstyle Id_a \otimes \mathcal{F}_2(b,c)} & & \downarrow{\scriptstyle \mathcal{F}_2(a,b) \otimes Id_c} \\
\mathcal{F}_0(a) \otimes \mathcal{F}_0(b \otimes c) & & \mathcal{F}_0(a \otimes b) \otimes \mathcal{F}_0(c) \\
\downarrow{\scriptstyle \mathcal{F}_2(a, b \otimes c)} & & \downarrow{\scriptstyle \mathcal{F}_2(a \otimes b, c)} \\
\mathcal{F}_0(a \otimes (b \otimes c)) & \xrightarrow{\mathcal{F}_0(\alpha_{a,b,c})} & \mathcal{F}_0((a \otimes b) \otimes c)
\end{array} \tag{4.7}$$

[1]For fusion categories there is some redundancy in this definition. It was mentioned in (Ostrik, 2003) that for semi-simple categories the existence of right duals implies the existence of left duals and vice versa. It was shown in (Davidovich et al., 2013) that the omitted right duality condition is implied by the first.

$$\begin{array}{ccc} \mathbf{1} \otimes \mathcal{F}_0(a) & \xrightarrow{\lambda_{\mathcal{F}_0(a)}} & \mathcal{F}_0(a) \\ \downarrow{\scriptstyle \mathcal{F}_1 \otimes Id_{\mathcal{F}_0(a)}} & & \uparrow{\scriptstyle \mathcal{F}_0(\lambda_a)} \\ \mathcal{F}(\mathbf{1}) \otimes \mathcal{F}_0(a) & \xrightarrow{\mathcal{F}_2(\mathbf{1},a)} & \mathcal{F}_0(\mathbf{1} \otimes a) \end{array} \tag{4.8}$$

$$\begin{array}{ccc} \mathcal{F}_0(a) \otimes \mathbf{1} & \xrightarrow{\rho_{\mathcal{F}_0(a)}} & \mathcal{F}_0(a) \\ \downarrow{\scriptstyle Id_{\mathcal{F}_0(a)} \otimes \mathcal{F}_1} & & \uparrow{\scriptstyle \mathcal{F}_0(\rho_a)} \\ \mathcal{F}_0(a) \otimes \mathcal{F}(\mathbf{1}) & \xrightarrow{\mathcal{F}_2(a,\mathbf{1})} & \mathcal{F}_0(a \otimes \mathbf{1}) \end{array} \tag{4.9}$$

which preserve the monoidal structure. Following (Kassel, 1995), we define that $\mathcal{C}$ and $\mathcal{D}$ are *monoidally equivalent* iff there exists a pair of monoidal functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{C} \to \mathcal{D}$ such that $\mathcal{F} \circ \mathcal{G}$ and $\mathcal{G} \circ \mathcal{F}$ are naturally monoidal isomorphic to the identity functors on $\mathcal{C}$ and $\mathcal{D}$ respectively.

Definition 4.1.4. A *Fusion Category* $\mathcal{C}$ over k is a k-linear semi-simple category that is also rigid monoidal with finitely many equivalence classes of simple objects such that $\mathcal{C}(\mathbf{1}, \mathbf{1}) \equiv k$. We denote by $\mathcal{K}_0(\mathcal{C})$ the Grothendieck ring of $\mathcal{C}$. This is a based ring as defined in 4.1.1 with basis elements corresponding to equivalence classes of simple objects $a, b, \ldots \in \mathcal{C}_0$ and multiplication induced from $\otimes$ via $N^b_{a_1 \ldots a_n} = dim(\mathcal{C}(a_1 \otimes \ldots \otimes a_n, b))$. We say that $\mathcal{C}$ is multiplicity free if $\mathcal{K}_0(\mathcal{C})$ is multiplicity free. Given two fusion categories $\mathcal{C}$ and $\mathcal{D}$ we say that they are *Grothendieck equivalent* if and only if $\mathcal{K}_0(\mathcal{C}) \cong \mathcal{K}_0(\mathcal{D})$.

We say that a based ring (R, B) *admits categorification* if there exists a fusion category $\mathcal{C}$ such that $(R, B) \cong \mathcal{K}_0(\mathcal{C})$. Important questions are then: (1) given a based ring (R, B) does it admit a categorification and (2) if so, how many?

Theorem 4.1.5 (Ocneanu rigidity)**.** *There are only finitely many fusion categories (up to monoidal equivalence) with a given Grothendieck ring $K_0(\mathcal{C})$.*

4.1.5 is proven in (Etingof and Ostrik, 2004) as Theorem 2.28.

4.1.1 Examples of Fusion Categories

All of the examples in Sections 2.3 and 3.4 define fusion categories. Those examples are also representatives for entire classes. Particularly, S_3, $\mathbb{Z}_2$, $Rep(S_3)$, and Toric code describe fusion categories coming directly from finite groups.

Fusion categories can be regarded as generalizations of these categories. They describe invariant structures under an algebra of symmetries which doesn't come from groups. The first example of where such symmetries arose was in the study of the 1+1D Hubbard model (See (Essler et al., 2005) for a pedagogical treatment). This led to the development of *quantum groups* whose representation categories are ribbon and modular categories.

Finite Groups

Recall that a group $(G, *)$ is a set G together with an associative binary operation $* : G \times G \to G$ and privileged element e such that for all $g \in G$, $g * e = e * g = g$ and there exists $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$. The multiplication $*$ defines a multiplication map as specified in (4.20) and so the structure coefficients are well defined. Indeed, an easy source of based rings is to extend a given group additively using $\mathbb{Z}$.

Fusion categories coming from G referred to as pointed, coming from the relationship with Hopf algebras, which we will not cover. They all arise when considering categories of finite dimensional vector spaces graded by G which may be twisted by a 3-cocycle in $H^3(G, \mathbb{C})$, generally referred to as Vec_G^ω. A three cocycle is a map $\omega : G \times G \times G \to \mathbb{C}^\times$ satisfying

$$\omega(a, b, c)\omega(a, b * c, d)\omega(b, c, d) = \omega(a, b, c * d)\omega(a * b, c, d).$$

This is clearly just the pentagon relation. As such pointed fusion categories associated to G are classified by $H^3(G, \mathbb{C}^\times)$ and the co-homology of finite groups is well understood. In this case, $\mathbb{Z}_2$, S_3, and Toric Code all define Grothendieck rings for pointed fusion categories.

Representation categories of finite groups

Finite group representations arise in the study of second order phase-transitions using Ginzburg-Landau theory and describe symmetries of a physical system (Landau, 1969; Toledano and Toledano, 1987). They also arise in crystallography and studies of molecules (Sternberg1994).

A complex representation (ρ, V) of a finite group G is a complex vector space V together with a map $\rho : G \to End(V)$. We say that (ρ, V) (or sometimes just V) is a finite dimensional representation if V is finite dimensional. A linear map $\gamma : V \to W$ between representations (V, ρ) and (W, ϕ) of G is called equivariant if $\gamma \circ \rho(g) = \phi(g) \circ \gamma$. A representation V is simple if its only sub-representations are $\{0\}$ and V and semi-simple if it is a direct sum of simple representations.

If (V, ρ) and (W, ϕ) are representations of G, then their tensor product as vector spaces $V \otimes W$ is a representation with $\rho \otimes \phi$. If V and W are *simple* representations, then their tensor product is a *semi-simple* representation and decomposes into a direct sum of simple ones. If $V_1, \ldots, V_n$ are the simple representations of G, then the number of times V_k appears in the tensor product decomposition of V_i and V_j may be written as N_{ij}^k and it is simple to see that these will satisfy exactly the conditions described in 4.1.1 where $\mathbf{1}$ is the one dimensional representation that sends all elements of G to 1 in $\mathbb{C}$. The category $Rep(G)$ is then the fusion category whose objects are simple representations $V_1, \ldots, V_n$ of G, morphisms are equivariant maps, and tensor product is the one above. As the name would imply $Rep(S_3)$ is the representation category of the group S_3.

4.1.2 Pivotal, Spherical, and Unitary Structures

To define physical systems, we are specifically interested in *unitary fusion categories*. To do so, we need some notions of dimension. Our first is at the level of based rings.

Definition 4.1.6. Let (R, B) be a based ring as in 4.1.1 and for $a \in R$, define the matrix N_a with rows and columns indexed by elements of R and whose entries are given by $(N_a)_{bc} = N_{ab}^c$. The *Frobenius-Perron dimension* $FP(a)$ of a is the largest positive real eigenvalue of N_a. The Frobenius-Perron dimension of (R, B) is $FP((R, B)) = \sum_{a \in R} FP(a)^2$.

To define a notion of dimension at the categorical level we first need the notion of a pivotal structure.

Definition 4.1.7. Let $\mathcal{C}$ be a fusion category and ϵ be a natural isomorphism $\epsilon : ** \to Id$ of the double dual and identity functors. For all equivalence classes of simple objects a of $\mathcal{C}$ this gives us a morphism $\epsilon_a : a^{**} \to a$. From this we can define the left and right *quantum trace* of a morphism f on a,

$$tr_l(f) = ev_a \circ (f \otimes Id_{a^*}) \circ ((\epsilon_a)^{-1} \otimes Id_{a^*}) \circ coev_{a^*} \tag{4.10}$$

$$tr_r(f) = ev_{a^*} \circ (Id_{a^*} \otimes \epsilon_a) \circ (Id_{a^*} \otimes f) \circ coev_a \tag{4.11}$$

where $ev_{_}$ and $coev_{_}$ are the evaluation and co-evaluation maps. A natural isomorphism ϵ such that $\epsilon_{a \otimes b} \cong \epsilon_a \otimes \epsilon_b$ is called a *pivotal isomorphism*. A *pivotal fusion category* is a fusion category equipped with a pivotal isomorphism.

Remark 4.1.8. If one draws the appropriate pictures for (4.10) and (4.11), you see that the ϵ map plays the same role that it does in (2.9) of section 2.2.

For every isomorphism class of simple objects a in a pivotal fusion category $\mathcal{C}$, one defines the left quantum dimension $q_l(a)$ and the right quantum dimension $q_r(a)$ as the quantum

traces of the identity morphism on a. A *spherical fusion category* is a pivotal fusion category such that the left and right quantum dimensions of all objects coincide. In this case we refer simply to the quantum dimension of an object a as q_a.

For a spherical fusion category $\mathcal{C}$ we define the matrix D to be the diagonal matrix with entries q_a for all equivalence classes a. We also define the *categorical quantum dimension* $q(\mathcal{C}) = \sum_a (q_a)^2$. A spherical fusion category is called *pseudo-unitary* iff $q(\mathcal{C}) = FP(\mathcal{K}_0(\mathcal{C}))$ and there exists a canonical choice of spherical structure such that $q_a = FP(a)$ for all a. We can now define unitary categories.

Definition 4.1.9. A *conjugation* on a fusion category is a family of conjugate linear maps $\mathcal{C}(x,y) \to \mathcal{C}(y,x)$, $f \to \bar{f}$ satisfying

$$\bar{\bar{f}} = f, \quad \overline{f \otimes g} = \bar{f} \otimes \bar{g}, \quad \text{and } \overline{f \circ g} = \bar{g} \circ \bar{f}.$$

A fusion category is *unitary* if $f = 0$ whenever $\bar{f} \circ f = 0$(Müger, 2003a). In this case then, all of the $\mathcal{C}(a \otimes b, c)$ spaces are Hilbert spaces (Kitaev, 2006). Unitary implies $q(a) = FP(a)$ and so then pseudo-unitary and spherical.

4.1.3 Modular Categories

Fusion categories do not capture all of the properties of anyons in a topological phase. Specifically, they do not contain any information about their mutual statistics. In order to do this, one needs the notion of a braided category, which one combines with a pivotal structure to obtain balanced, ribbon, and modular categories.

Definition 4.1.10. Let $\mathcal{C}$ be a fusion category. We say that $\mathcal{C}$ is braided if there exist natural isomorphisms $\beta_{x,y} : y \otimes x \to x \otimes y$ satisfying

$$\begin{array}{ccccc} & & (b\otimes c)\otimes a & \xrightarrow{\beta_{a,(b\otimes c)}} & a\otimes(b\otimes c) & & \\ & \overset{\alpha_{b,c,a}}{\nearrow} & & & & \overset{\alpha_{a,b,c}}{\searrow} & \\ b\otimes(c\otimes a) & & & & & & (a\otimes b)\otimes c \\ & \underset{\beta_{a,c}}{\searrow} & & & & \underset{\beta_{a,b}}{\nearrow} & \\ & & b\otimes(a\otimes c) & \xrightarrow[\alpha_{b,a,c}]{} & (b\otimes a)c & & \end{array} \tag{4.12}$$

Definition 4.1.11. If $\mathcal{C}$ and $\mathcal{D}$ are braided monoidal categories then a *braided monoidal functor* is a monoidal functor as defined in 4.1.3 which must satisfy the additional condition

$$\begin{array}{ccc} \mathcal{F}_0(b)\otimes\mathcal{F}_0(a) & \xrightarrow{\mathcal{F}_2(b,a)} & \mathcal{F}_0(b\otimes a) \\ \Big\downarrow{\scriptstyle c_{\mathcal{F}_0(a),\mathcal{F}_0(b)}} & & \Big\downarrow{\scriptstyle \mathcal{F}_0(c_{a,b})} \\ \mathcal{F}_0(a)\otimes\mathcal{F}_0(b) & \xrightarrow{\mathcal{F}_2(a,b)} & \mathcal{F}_0(a\otimes b) \end{array} \tag{4.13}$$

which preserves the braiding. If $\mathcal{C}$ and $\mathcal{D}$ are braided monoidal, then they are *braided monoidally equivalent* if $\mathcal{F}$ and $\mathcal{G}$ are braided monoidal functors.

For braided fusion categories there is a canonical natural isomorphism

$$\psi_a = \rho_{a^{**}} \circ (Id_{a^{**}} \otimes ev_a) \circ \alpha^{-1}_{a^{**},a,a^*} \circ (c_{a,a^{**}} \otimes Id^*_a) \circ \alpha_{a,a^{**},a^*} \circ (Id_a \otimes coev_{a^*}) \circ \rho_a^{-1} \tag{4.14}$$

and a *balanced fusion category* $\mathcal{M}$ is a braided pivotal fusion category with balancing $\theta_a = \epsilon_a \circ \psi_a$. The trace of this morphism, by abuse of notation, will also be denoted θ_a and the diagonal matrix with entries θ_a will be called the T-matrix. A *ribbon fusion category* is a balanced fusion category that is also spherical.

There also exist a family of invariants S_{ab} corresponding to the evaluations of Hopf links colored by each pair of equivalence classes (a, b). The matrix with entries S_{ab} is called

the S-Matrix and a *modular category* is a ribbon fusion category with non-degenerate S-matrix. In the case that $\mathcal{C}$ is modular than all θ_a are roots of unity. The matrices $\frac{1}{q(\mathcal{C})}S$ and $T = diag(\theta_a)$ give a representation of the modular group $SL(2,\mathbb{Z})$. It is believed that the pair (S,T) uniquely determine the modular category(Bruillard et al., 2016) and typically what one does in classifying modular categories (e.g., (Rowell et al., 2009),(Bruillard et al., 2016)) is enumerate the admissible pairs (S,T).

4.1.4 The Monoidal Center of a Unitary Fusion Category

While every modular category is a fusion category, the converse is not true. However, given a unitary fusion category $\mathcal{C}$, we can always construct a unitary modular category $\mathcal{Z}(\mathcal{C})$ called the *monoidal center*.

Definition 4.1.12. Given a unitary fusion category $\mathcal{C}$, its *monoidal center* $\mathcal{Z}(\mathcal{C})$ is modular category such that

1. objects in $\mathcal{Z}(\mathcal{C})$ are pairs (X, e_X) where $X \in \mathcal{C}_0$ and e_X is a half-braiding as defined in (2.16),

2. There exists a (braided) monoidal functor taking $\mathcal{Z}(\mathcal{C})$ to $\mathcal{C}$ such that $\mathcal{F}_0((X, e_x)) \to X$.

An important theorem of (Müger, 2003b) says that if $\mathcal{C}$ is spherical then $\mathcal{Z}(\mathcal{C})$ is modular. If $\mathcal{C}$ is modular to begin with, then $\mathcal{Z}(\mathcal{C}) \cong \mathcal{C} \boxtimes \bar{\mathcal{C}}$ where $\boxtimes$ is the Deligne product(Kassel, 1995) and $\bar{\mathcal{C}}$ is the category with the same objects but uses the inverse braiding. Given a string-net $(\mathcal{C}, \Sigma)$, it is precisely the objects of $\mathcal{Z}(\mathcal{C})$ which correspond to its excitations.

4.2 Fusion and Modular Systems

An important question is how one goes about constructing fusion categories. One way to do so(Davidovich et al., 2013) is through a collection of numbers called a fusion system. In general this process is referred to as *skeletalization*.

Definition 4.2.1. Fix a based ring (R, B). For every $(a, b, c, d) \in B^{\times 4}$, such that $N_{abc}^d \neq 0$ define the matrix $\Phi_{abc}^d \in Mat_{N_{abc}^d \times N_{abc}^d}(k)$ with entries denoted

$$\Phi_{abc}^d \left[\begin{smallmatrix} i' & f & j' \\ i & e & j \end{smallmatrix}\right], \quad \begin{matrix} f \in B \\ e \in B \end{matrix}, \quad \begin{matrix} i' \in \{1, \ldots, N_{af}^d\} \\ i \in \{1, \ldots, N_{ab}^e\} \end{matrix}, \quad \begin{matrix} j' \in \{1, \ldots, N_{bc}^f\} \\ j \in \{1, \ldots, N_{ec}^d\} \end{matrix}$$

and define $\Phi(R, B)$ to be the set of entries in all Φ_{abc}^d. The ideal in $k[\Phi(R, B)]$ generated by

$$\Phi_{a\mathbf{1}b}^c = Id_{N_{ab}^c} \tag{4.15}$$

$$\Phi_{aa^*a}^a \left[\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix}\right] \neq 0 \tag{4.16}$$

$$\sum_{n''=1}^{N_{fh}^e} \Phi_{fcd}^e \left[\begin{smallmatrix} l'' & h & n'' \\ m & i & n \end{smallmatrix}\right] \Phi_{abh}^e \left[\begin{smallmatrix} m''' & j & n' \\ l & f & n'' \end{smallmatrix}\right] = \tag{4.17}$$

$$\sum_{g \in L} \sum_{l'=1}^{N_{bc}^g} \sum_{m'=1}^{N_{ag}^i} \sum_{m''=1}^{N_{gd}^j} \Phi_{abc}^d \left[\begin{smallmatrix} l' & g & m' \\ l & f & m \end{smallmatrix}\right] \Phi_{agd}^e \left[\begin{smallmatrix} m'' & j & n'' \\ m' & i & n' \end{smallmatrix}\right] \Phi_{bcd}^j \left[\begin{smallmatrix} l'' & h & m''' \\ l' & g & m'' \end{smallmatrix}\right]$$

together with the condition that the Φ_{abc}^d be invertible defines an algebraic scheme $X(R, B)$ which we call a *fusion scheme*. A point $\mathbf{F} \in X(R, B)$ is a map $F : \Phi(R, B) \to k$ such that the equations defined by (4.15)-(4.17) are satisfied. We will fix the notation $F_{abc}^d \left[\begin{smallmatrix} i' & f & j' \\ i & e & j \end{smallmatrix}\right] := F(\Phi_{abc}^d \left[\begin{smallmatrix} i' & f & j' \\ i & e & j \end{smallmatrix}\right])$ and F_{abc}^d for the matrix with entries $F_{abc}^d \left[\begin{smallmatrix} i' & f & j' \\ i & e & j \end{smallmatrix}\right]$. The triple (B, N, F) is called a *fusion system*.

The following is central to the content of this dissertation:

Theorem 4.2.2 (Reconstruction)**.**

1. *Given a fusion category $\mathcal{C}$ over $\mathbb{C}$, a fusion system over $\mathbb{C}$, denoted $(B, N, F)(\mathcal{C})$, can be extracted.*

2. *Given a fusion system (B, N, F), one can construct a fusion category $\mathcal{C}(B, N, F)$.*

3. *Given $\mathcal{C}$, $\mathcal{C}((B, N, F)(\mathcal{C})) \cong \mathcal{C}$ as fusion categories.*

4.2.2 is proven in (Davidovich et al., 2013) as Proposition 3.7, though a similar theorem also appears in (Yamagami, 2002).

Corollary 4.2.3. *Two fusion categories $\mathcal{C}$ and $\mathcal{D}$ are Grothendieck equivalent iff given any two fusion systems (B, N, F) and (B', N', F') extracted from them there exists a bijection $f : L \to L'$ such that for all $\nu \in Aut(N)$ there exists $\nu' \in Aut(N')$ such that $f \circ \nu = \nu' \circ f$.*

Loosely, what 4.2.2 does is validate our (careful) movement back and forth between the fusion categories and fusion systems. A more immediate consequence of this though is that a string-net model can be redefined as a UFC $\mathcal{C}$ together with a triangulation Δ on a closed manifold Σ. However, this is also exactly the data used to define the so called Turaev-Viro TQFTs (originally from (Turaev and Viro, 1992a), also see (Bakalov and Kirillov, 2001) for a review). That String-net models are a Hamiltonian realisation of Turaev-Viro TQFTs is given in (Kirillov and Balsam, 2010). The important consequence of this however is that the triangulation of Σ chosen is irrelevant. Thus, a String-net model can be regarded as the pair $(\mathcal{C}, \Sigma)$.

Remark 4.2.4. All of the data for examples already provided – $\mathbb{Z}_2$, Toric Code, Fibonacci, $D(\mathbf{Fib})$, S_3, and $Rep(D(S_3))$ – are examples of fusion systems.

There also exists arithmetic data for pivotal structures.

Definition 4.2.5. Let $X(R, B)$ be a fusion scheme with coordinate ring CR and consider the polynomial ring $CR[H(B)], H(B) = \{\eta_a | a \in B\}$ over CR. The ideal in this generated by

$$\eta_c^{-1}\eta_a\eta_b = \sum_{s=1}^{N_{bc^*}^{a^*}} \sum_{t=1}^{N_{c^*a}^{b^*}} \Phi^{\mathbf{1}}_{abc^*} \left[\begin{smallmatrix} i & c & 1 \\ 1 & a^* & s \end{smallmatrix}\right] \Phi^{\mathbf{1}}_{bc^*a} \left[\begin{smallmatrix} s & a^* & 1 \\ 1 & b^* & t \end{smallmatrix}\right] \Phi^{\mathbf{1}}_{bc^*a} \left[\begin{smallmatrix} t & b^* & 1 \\ 1 & c & i \end{smallmatrix}\right] \tag{4.18}$$

for $i \in \{1, \ldots, N_{ab}^c\}$ defines another algebraic scheme $X(R, B, H)$ which will be called a *pivotal extension* of $X(R, B)$. A point (F, ϵ) defines a *pair* of maps such that (B, N, F) is a

fusion system and $\epsilon : H(B) \to \mathbb{C}$ such that the equations defined by (4.18) are satisfied. We fix the notation $\epsilon_a := \epsilon(\eta_a)$ and call the set $\{\epsilon_a | a \in B\}$ *pivotal coefficients* for F and define the set (B, N, F, ϵ) to be a *pivotal system*.

Ocneanu rigidity guarantees that for a fusion system (B, N, F) there are only finitely many maps ϵ satisfying (4.18). In (Davidovich et al., 2013, Proposition 3.12) we have an analogue to theorem 4.2.2: Given a pivotal system (B, N, F, ϵ), ϵ can be used to define a pivotal structure on $\mathcal{C}(B, N, F)$ and vice versa. If $(B, N, F)(\mathcal{C})$ is a fusion system extracted from $\mathcal{C}$, then pivotal structures on $\mathcal{C}$ are in 1-1 correspondence with maps ϵ compatible with F.

Given a fusion category $\mathcal{C}$ with pivotal structure ϵ, if is a spherical structure then there exists a fusion system $(L, N, F)(\mathcal{C})$ such that all $\epsilon_a = \pm 1$. By skeletalizing the rigidity conditions and using the choice of basis as in (Davidovich et al., 2013, Lemma 3.4), the quantum dimensions can be computed from F and ϵ as

$$q_l(a) = \epsilon_a (F^{a^*}_{a^* a a^*} \left[\begin{smallmatrix} 1 & \mathbf{1} \\ \mathbf{1} & 1 \end{smallmatrix}\right])^{-1} \text{ and } \qquad q_r(a) = (\epsilon_a F^{a}_{a a^* a} \left[\begin{smallmatrix} 1 & \mathbf{1} \\ \mathbf{1} & 1 \end{smallmatrix}\right])^{-1} \tag{4.19}$$

If $\mathcal{C}$ admits the structure of a unitary fusion category, then there exists a fusion system $(B, N, F)(\mathcal{C})$ such that all F^d_{abc} matrices are unitary. Conversely (See (Tambara and Yamagami, 1998), Section 4), if given (L, N) there exists a solution to (4.15)-(4.17) such that all F^d_{abc} matrices are unitary, then $\mathcal{C}(L, N, F)$ admits a unitary structure.

Definition 4.2.6. Fix a fusion scheme $X(R, B)$ such that for all $a, b, c \in B, N^c_{ab} = N^c_{ba}$ with coordinate ring CR. For every $(a, b, c) \in B^{\times 3}$, such that $N^c_{ab} \neq 0$ define the matrices $\mathrm{P}^c_{ab} \in Mat_{N^c_{ab} \times N^c_{ab}}(k)$ and $\mathrm{Y}^c_{ab} \in Mat_{N^c_{ab} \times N^c_{ab}}(k)$ with entries denoted

$$\mathrm{P}^c_{ab} \left[\begin{smallmatrix} i' \\ i \end{smallmatrix}\right], i, i' \in \{1, \ldots, N^c_{ab}\} \text{ and } \mathrm{Y}^c_{ab} \left[\begin{smallmatrix} i' \\ i \end{smallmatrix}\right], i, i' \in \{1, \ldots, N^c_{ab}\}.$$

Define $\mathrm{P}(R, B)$ and $\mathrm{Y}(R, B)$ to be the set of entries in all P^c_{ab} and Y^c_{ab} respectively. The ideal in the polynomial ring $CR[\mathrm{P}(R, B) \cup \mathrm{Y}(R, B)]$ generated by

$$\mathrm{Y}^c_{ab} \mathrm{P}^c_{ab} = Id_{N^c_{ab}} \tag{4.20}$$

$$\sum_{i'=1}^{N_{ac}^{d}}\sum_{j'=1}^{N_{bc}^{d'}} \mathrm{P}_{ac}^{g}\left[\begin{smallmatrix} i' \\ i \end{smallmatrix}\right] \Phi_{acb}^{d}\left[\begin{smallmatrix} i'' & f & j' \\ i' & g & j \end{smallmatrix}\right] \mathrm{P}_{bc}^{f}\left[\begin{smallmatrix} j'' \\ j' \end{smallmatrix}\right] =$$

$$\sum_{e\in L}\sum_{k=1}^{N_{ce}^{d}}\sum_{k'=1}^{N_{ec}^{d}}\sum_{l=1}^{N_{ab}^{e}} \Phi_{cab}^{d}\left[\begin{smallmatrix} k & e & l \\ i & g & j \end{smallmatrix}\right] \mathrm{P}_{ec}^{d}\left[\begin{smallmatrix} k' \\ k \end{smallmatrix}\right] \Phi_{abc}^{d}\left[\begin{smallmatrix} i' & f & j' \\ l & e & k' \end{smallmatrix}\right] \tag{4.21}$$

$$\sum_{i'=1}^{N_{ac}^{d}}\sum_{j'=1}^{N_{bc}^{d'}} \Upsilon_{ac}^{g}\left[\begin{smallmatrix} i' \\ i \end{smallmatrix}\right] \Phi_{acb}^{d}\left[\begin{smallmatrix} i'' & f & j' \\ i' & g & j \end{smallmatrix}\right] \Upsilon_{bc}^{f}\left[\begin{smallmatrix} j'' \\ j' \end{smallmatrix}\right] =$$

$$\sum_{e\in L}\sum_{k=1}^{N_{ce}^{d}}\sum_{k'=1}^{N_{ec}^{d}}\sum_{l=1}^{N_{ab}^{e}} \Phi_{cab}^{d}\left[\begin{smallmatrix} k & e & l \\ i & g & j \end{smallmatrix}\right] \Upsilon_{ec}^{d}\left[\begin{smallmatrix} k' \\ k \end{smallmatrix}\right] \Phi_{abc}^{d}\left[\begin{smallmatrix} i' & f & j' \\ l & e & k' \end{smallmatrix}\right] \tag{4.22}$$

define another algebraic scheme $X(\mathcal{R}, B, \mathrm{P})$ which will be called a *braided extension* of $X(R, B)$. A point (F, R) defines a *pair* of maps such that (B, N, F) is a fusion system and $\mathbb{R} : \mathrm{P}(R, B) \to \mathbb{C}$ such that the equations defined by (4.20)-(4.22) are satisfied. We fix the notation $R_{ab}^{c}\left[\begin{smallmatrix} i' \\ i \end{smallmatrix}\right] := R(P_{ab}^{c}\left[\begin{smallmatrix} i' \\ i \end{smallmatrix}\right])$ and call the set (B, N, F, R) a *braided system*.

Remark 4.2.7. There are several important things to note here.

1. Ocneanu rigidity again guarantees that for a fusion system (B, N, F) there are only finitely many maps R such that (4.20)-(4.22) are satisfied. This is again part of (Davidovich et al., 2013, Proposition 3.12) from which we also have reconstruction theorems.

2. Both (4.21) and (4.22) are necessary to for constructing a braiding. However, it is possible to construct solutions to only one of (4.21) or (4.22) without solutions to the other(Hagge and Hong, 2009). As a counter example, the category $\frac{1}{2}E_6$ corresponding to the even half of the E_6-subfactor is not braided. Equations (4.21) or (4.22) can be solved, but not both simultaneously.

3. A point in $X(R, B, \mathrm{P})$ also defines a map $Q : \Upsilon(R, B) \to \mathbb{C}$. Given a fusion system (B, N, F), both are needed to construct a braiding on $\mathcal{C}(B, N, F)$. However, given a pair (R, Q) satisfying (4.20)-(4.22), (4.20) guarantees that the map Q is uniquely determined by specifying R.

4. If the pair of maps (R, Q) defines a braiding, which is to say that R satisfies (4.21) and Q satisfies (4.22), then (Q, R) also defines a braiding. The choice of which map is which, however, can define in-equivalent braidings on a fusion category $\mathcal{C}(B, N, F)$.

Proposition 4.2.8. *Let $X(R, B)$ be a fusion scheme which admits a non-empty pivotal extension $X(R, B, H)$ and a non-empty braided extension $X(R, B, P)$. Then $X(R, B)$ admits both simultaneously.*

Proof. This follows from Hilbert's basis theorem and the fact that the variables and relations for one are independent of the variables and relations for the other. □

Consider a fusion scheme with braided extension $X(R, B, \mathrm{P})$ and, for simplicity, let Ψ^d_{abc} be the matrix whose entries are rational functions corresponding to the entries of $(\Phi^d_{abc})^{-1}$. Then the rational functions

$$\hat{S}_{ab} = \sum_{c\in B} \sum_{i=1}^{N^c_{ab^*}} \sum_{j=1}^{N^a_{cb}} \sum_{i'=1}^{N^c_{b^*a}} \sum_{i''=1}^{N^c_{ab^*}} \Psi^a_{ab^*b} \left[\begin{smallmatrix} i & c & j \\ 1 & \mathbf{1} & 1 \end{smallmatrix}\right] \mathrm{P}^c_{b^*a} \left[\begin{smallmatrix} i' \\ i \end{smallmatrix}\right] \mathrm{P}^c_{ab^*} \left[\begin{smallmatrix} i'' \\ i' \end{smallmatrix}\right] \Phi^a_{ab^*b} \left[\begin{smallmatrix} 1 & \mathbf{1} & 1 \\ i'' & c & j \end{smallmatrix}\right] \tag{4.23}$$

indexed by all pairs (a, b) of objects in B are well defined and, importantly, non-singular on all of $X(R, B, \mathrm{P})$. Together these define a matrix $\hat{S}$.

Definition 4.2.9. Fix a based ring (R, B). A *modular scheme* $X(R, B, P, H)$ is an algebraic scheme such that $X(R, B)$ is a fusion scheme, $X(R, B, H)$ is a pivotal extension, and $X(R, B, P)$ is a braided extension, and the matrix $\hat{S}$ defined in (4.23) is non-degenerate. A point (F, R, ϵ) defines a triple of maps such that (B, N, F) is a fusion system, (B, N, F, ϵ) is a pivotal system, and (B, N, F, R) is a braided system. We will call the 5-tuple (B, N, F, R, ϵ) a *modular system*.

Remark 4.2.10. The matrix S and the matrix $\hat{S}$ are related via $S = D\hat{S}D$ where D is the diagonal matrix whose entries are the quantum dimensions (Davidovich et al., 2013).

The last bit of arithmetic data for a modular category we need to be able to compute is the T matrix. Given a modular system (B, N, F, R, ϵ), it was obtained in (Kitaev, 2006) that the twists θ_a can be expressed as

$$\theta_a = (q_a)^{-1} \sum_{c \in L} \sum_{i=1}^{N_{aa}^c} q_c R_{aa}^c \left[\begin{smallmatrix} i \\ i \end{smallmatrix} \right]. \tag{4.24}$$

With this we can now construct all of the data about fusion through modular categories that will occur from here on out.

CHAPTER 5

GEOMETRIC INVARIANTS FOR FUSION CATEGORIES

In this chapter we will be concerned with determining whether or not two distinct fusion (modular) systems give rise to the same topological phase. In the language of category theory, this means we are concerned with the question of determining whether or not the associated fusion categories are monoidally equivalent. To that end, the author and T. Hagge developed in (Hagge and Titsworth, 2015) a construction for geometric invariants which are capable of distinguishing arbitrary equivalence classes. The whole thing is rooted in Mumford's Geometric Invariant Theory(Mumford et al., 1994) which is a geometric version of Hilbert's results on computing rings of invariants. This is the real reason behind all of the algebro-geometric machinery above. By considering the algebraic schemes[1] we are able to cast the problem in the correct light so that the solution becomes immediate.

The following text is taken mostly verbatim from (Hagge and Titsworth, 2015). Section 2 from that work is omitted since it appears in the previous chapter.

5.1 Introduction

A fundamental problem in the study of fusion categories(Etingof et al., 2005) is to classify all fusion categories over a field k with Grothendieck ring isomorphic to a based ring (R, B) up to monoidal equivalence. This is extremely difficult in general; a classification of all fusion categories would necessarily include a classification of all finite groups. It is, however, sufficient to solve a set of polynomial equations known as the *pentagon equations*.

[1]More classically, one could consider the associated algebraic sets, which is to say one could consider the radical of the ideals in (4.15)-(4.17),(4.18), and (4.20)-(4.22) which would make $X(R, B)$ reduced. However, this gets us absolutely nothing and so we omit the unnecessary assumption.

There are a number of practical difficulties to this approach. First, the number of equations and variables involved tends to grow exponentially as the number of simple objects increases. Gröbner basis methods for solving such systems have worst-case complexity which is double-exponential in the size of the input. Second, although Ocneanu rigidity (see (Etingof et al., 2005)) guarantees that there are only a finite number of equivalence classes of fusion categories with a given Grothendieck ring, solutions are only unique up to gauge transformations given by a group G and automorphisms of the Grothendieck ring given by a group $Aut(R, B)$. Thus it is often necessary to check solutions for equivalence.

In the presence of extra structure, there are a number of techniques to make classification easier. For example, pointed categories with Grothendieck ring determined by a finite group G are classified by $H^3(G, k^\times)$. Every unitary fusion category comes from a finite index subfactor(Popa, 1990). Certain fusion categories can be realized using Cuntz algebras(Evans and Gannon, 2014) or Leavitt algebras(Evans and Gannon, 2015). Monoidal algebras(Kazhdan and Wenzl, 1993; Tuba and Wenzl, 2005) have been used to classify all fusion categories coming from A-series Lie algebras and braided fusion categories coming from B, C, and D-series Lie algebras.

Fusion categories are often classified using invariants. In (Basak and Johnson, 2014) it is shown that two Tambara-Yamagami categories(Tambara and Yamagami, 1998) $\mathcal{C}$ and $\mathcal{C}'$ are equivalent if all of their higher Frobenius-Schur indicators as introduced by (Ng and Schauenburg, 2007) are equal. Categorical invariants such as the Turaev-Viro(Turaev and Viro, 1992b) and Reshetikin-Turaev(Reshetikhin and Turaev, 1990, 1991b) invariants become invariants of (subclasses of) fusion categories when the manifold is fixed and the category is taken as a parameter. It is conjectured that the (S, T) matrices classify all modular categories (Bruillard et al., 2016).

In this work, we develop a new class of invariants strong enough to classify arbitrary fusion categories. These invariants are computable and for a large class of fusion categories we show

that our method leads to fast algorithms for computing gauge classes, monoidal classes and the Galois action on these classes. For a subset of these, we show how to determine values of invariants without first solving the pentagon equations.

Our approach uses the machinery of geometric invariant theory (see (Mumford et al., 1994) or (Dolgachev, 2003) for an introduction). Specifically, given a based ring (R, B), the pentagon equations generate an ideal in some polynomial k-algebra $k[\Phi(R, B)]$. From this, one can construct an algebraic scheme[2] $X(R, B)$ (see Definition 4.2.1 and Remark 5.2.2) which is acted upon by gauge group G (Definition 5.2.1) and Grothendieck ring automorphism group $Aut(R, B)$ (Definition 5.2.8). Theorems of (Mumford et al., 1994) then reduce the problem of determining gauge and monoidal classes to one of constructing regular functions on $X(R, B)$ invariant under the actions of G and $Aut(R, B)$. In particular, for a non-empty scheme $X(R, B)$ associated to (R, B) we prove the following:

Theorem 5.1.1. *Fix a based ring (R, B). Define $\mathscr{N}$ to be the number of gauge equivalence classes for fusion categories Grothendieck equivalent to (R, B) and $\mathscr{M}$ the number of monoidal equivalence classes. For all $\mathbf{F} \in X(R, B)$, denote by $\mathcal{C}(\mathbf{F})$ its associated fusion category. Then there exists $\mathscr{N} - 1$ G-invariant regular functions $f_1, \ldots, f_{\mathscr{N}-1}$ on $X(R, B)$ such that for all $\mathbf{F}, \mathbf{F}' \in X(R, B)$, $\mathcal{C}(\mathbf{F})$ and $\mathcal{C}(\mathbf{F}')$ are gauge equivalent if and only if*

$$f_i(\mathbf{F}') = f_i(\mathbf{F}), i = 1, \ldots, \mathscr{N} - 1 \tag{5.1}$$

Similarly, there exist $\mathscr{M} - 1$ $Aut(R, B) \ltimes G$-invariant regular functions $g_1, \ldots, g_{\mathscr{M}-1}$ on $X(R, B)$ such that for all $\mathbf{F}, \mathbf{F}' \in X(R, B)$, $\mathcal{C}(\mathbf{F})$ and $\mathbf{F}'$ are monoidally equivalent if and only if

$$g_j(\mathbf{F}') = g_j(\mathbf{F}), j = 1, \ldots, \mathscr{M} - 1 \tag{5.2}$$

[2] A separated affine scheme of finite type over $spec(k)$. These will be defined from affine algebraic sets, and by abuse of notation will also be denoted $X(R, B)$. These sets will themselves be useful, but the structure sheaf will carry all of the important information.

For the case of multiplicity free fusion categories (i.e., all structure coefficients for the Grothendieck ring are 0 or 1) we obtain a stronger statement:

Theorem 5.1.2. *Fix a based ring (R, B). Define $\mathscr{N}$ to be the number of gauge equivalence classes for fusion categories Grothendieck equivalent to (R, B) and $\mathscr{M}$ the number of monoidal equivalence classes. Then there exist*

$$P \leq \binom{\mathscr{N}}{2}$$

G-invariant rational monomials

$$m_i = (\phi_{i,1})^{k_{i,1}} \dots (\phi_{i,j})^{k_{i,j}}, \qquad \text{with} \qquad \phi \in \Phi(R, B) \qquad \text{and} \qquad k \in \mathbb{Z} \tag{5.3}$$

such that for all $\mathbf{F}, \mathbf{F}' \in X(R, B)$, $\mathcal{C}(\mathbf{F})$ and $\mathcal{C}(\mathbf{F}')$ are gauge equivalent if and only if

$$F(m_i) = F'(m_i), \qquad i = 1, \dots P \tag{5.4}$$

or both are simultaneously undefined. Similarly, there exist

$$Q \leq \binom{\mathscr{M}}{2}$$

$Aut(R, B)$-invariant linear combinations $l_1, \dots, l_Q$ of G-invariant rational monomials as in (5.3) such that $\mathcal{C}(\mathbf{F})$ and $\mathcal{C}(\mathbf{F}')$ are monoidally equivalent if and only if

$$F(l_j) = F'(l_j), \qquad j = 1, \dots Q. \tag{5.5}$$

Computing these quantities involves a technical difficulty. G-invariant rational monomials form a (finitely generated free Abelian) group. Naively, one would like to compute a basis for this group and evaluate $\mathbf{F} \in X(R, B)$ on the basis monomials. Unfortunately, if 0 is in the image of F the evaluation map on monomials is not totally defined and does not satisfy the homomorphism property. However, the sets $F^{-1}(0)$ are gauge invariant and given $F^{-1}(0)$ there exists a fast algorithm for computing a set of nonzero invariant monomials which uniquely determine the evaluations on all other monomials.[3]

[3] A priori, only the non-zero evaluations are determined, but as shown in Appendix 5.5 all other evaluations are determined by the non-zero ones.

Theorem 5.1.3. *Let (R, B) be multiplicity free and $F^{-1}(0)$ be the pre-image of 0 for some $\mathbf{F} \in X(R, B)$. Then for all $\mathbf{F}' \in X(R, B)$ with $(F')^{-1}(0) = F^{-1}(0)$, the evaluations of $\mathbf{F}'$ on all G-invariant rational monomials are completely determined by the non-zero evaluations of $\mathbf{F}'$ on a finite set $\mathcal{B}$. Given (R, B) and $F^{-1}(0)$ a choice of $\mathcal{B}$ is computable with worst case runtime complexity(Micciancio and Warinschi, 2001)*

$$\mathcal{O}(|\Phi||\Gamma|^4 \log^2(\mathcal{M})).$$

Together with Theorem 5.1.2 one obtains

Corollary 5.1.4. *Let $\mathbf{F}, \mathbf{F}' \in X(R, B)$ and $\rho \in Aut(R, B)$ such that $(\rho \cdot (F')^{-1}(0)) = F^{-1}(0)$. $\mathcal{C}(\mathbf{F})$ and $\mathcal{C}(\mathbf{F}')$ are then monoidally equivalent if and only if*

$$F(m) = F'(\rho \cdot m) \qquad \forall m \in \mathcal{B}. \tag{5.6}$$

This falls short of an algorithmic proof of Theorem 1.2, but for practical purposes is just as good. All of Theorem 5.1.1 through Corollary 5.1.4 are consequences of the following:

Theorem 5.1.5. *Let $X(R, B)$ be a fusion scheme acted upon by a gauge group G and automorphism group $Aut(R, B)$. Then there exists an algebraic scheme $Y(R, B)$ and surjective morphism of schemes*

$$\psi : X(R, B) \to Y(R, B)$$

such that $(Y(R, B), \psi)$ is a geometric quotient of $X(R, B)$ by G in the sense of (Mumford et al., 1994, Definition 0.6). Similarly, there exists an algebraic scheme $Z(R, B)$ and surjective morphism of schemes

$$\eta : Y(R, B) \to Z(R, B)$$

such that $(Z(R, B), \psi \circ \eta)$ is a geometric quotient of $X(R, B)$ by $Aut \ltimes G$.

The structure of this paper is as follows: In section 5.2, the algebraic groups G and $Aut(R,B) \ltimes G$ are defined, fusion schemes are defined, and Theorems 5.1.1, 5.1.5, and 5.1.2 are proven. Section 5.3 addresses computation issues, proves Theorem 5.1.3, and an algorithm is given for computing a classifying set of invariants. Finally, Section 5.4 provides concrete examples. The quotient scheme associated to the Fibonacci and Yang-Lee categories is constructed in Section 5.4.1. As a more significant example, in section 5.4.2 solutions the pentagon equations for the Grothendieck ring of $Rep(D(S_3))$ are used to compute monoidal classes and show the existence of a new fusion category which does not admit braiding.

Remark 5.1.6. While pivotal structures and braidings are ignored in the current work, the results presented carry over; the braiding and pivotal constants are also acted upon algebraically by G and there are in fact invariant functions which distinguish braided and pivotal equivalences classes of fusion categories. The argument follows lines similar to those given in this document and is omitted for brevity. Still, in presenting monoidal classes for $Rep(D(S_3))$ it is worth while to note which modular structures are associated to the underlying fusion categories, so this is done.

Remark 5.1.7. The solutions to pentagon equations for $Rep(D(S_3))$ are rather large and unwieldy, so we have opted to instead include them in a Mathematica package designed to manipulate the arithmetic data for fusion categories. This package can be found at http://mktitsworth.com/fusion-categories and includes example data for many other categories as well.

5.2 Invariants of Fusion Schemes

For brevity, fix a based ring (R,B) and let $X = X(R,B)$, $\Gamma = \Gamma(R,B)$, and $\Phi = \Phi(R,B)$.

5.2.1 Group Actions and Quotients

Definition 5.2.1. Given X, let $G = \prod_{\gamma^c_{ab} \in \Gamma} GL_{N^c_{ab}}(k)$. For $g \in G$, fix the notations

$$g^c_{ab} \in GL_{N^c_{ab}}(k) \text{ and } g^{ab}_c = (g^c_{ab})^{-1}.$$

Then

$$g = \prod_{\gamma^c_{ab} \in \Gamma} (g^c_{ab}) \text{ and } g^{-1} = \prod_{\gamma^c_{ab} \in \Gamma} (g^{ab}_c).$$

Let G act algebraically on the point $\mathbf{F} \in X$ such that for $g \cdot \mathbf{F}$, the induced map $(g \cdot F)$ is given by

$$(g \cdot F)^d_{abc} \left[\begin{smallmatrix} k' & f & l' \\ i' & e & j' \end{smallmatrix} \right] := \sum_{i=1}^{N^e_{ab}} \sum_{j=1}^{N^d_{ec}} \sum_{k=1}^{N^f_{bc}} \sum_{l=1}^{N^d_{af}} (g^e_{ab})^{i'}_i (g^d_{ec})^{j'}_j F^d_{abc} \left[\begin{smallmatrix} k & f & l \\ i & e & j \end{smallmatrix} \right] (g^{af}_d)^l_{l'} (g^{bc}_f)^k_{k'}. \tag{5.7}$$

It is straight forward to check that $g \cdot \mathbf{F}$ is a point of X. Call $\mathbf{F}, \mathbf{F}' \in X$ *gauge equivalent* and write $\mathbf{F} \sim \mathbf{F}'$ if they belong to the same orbit $G \cdot \mathbf{F}$ of the G-action. Say that $\mathcal{C}(\mathbf{F})$ and $\mathcal{C}(\mathbf{F}')$ are gauge equivalent if and only if $\mathbf{F} \sim \mathbf{F}'$.

Remark 5.2.2. For any algebraic set X associated to the zero set of a polynomial I, the *ring of regular functions* (with respect to I) on X is

$$O_X(X) := k[x_1, ..., x_n]/I.$$

There is a natural surjection from $O_X(X)$ to restrictions of polynomial functions on k^n to X and this is a bijection if I is a radical ideal. Given any two points in an algebraic set X, there is a polynomial (and thus a regular function) which distinguishes them.

One wishes to imitate this behavior at the level of orbits, that is, to find polynomial functions which distinguish the orbits of X under the action of the gauge group G and later the permutation group $Aut(R, B)$ as well. Let $O_X(X)^G$ be the ring of G-invariant regular functions on X, that is equivalence classes of G-invariant polynomial functions defined on X.

The desired polynomial functions must then be equivalence class representatives of elements of $O_X(X)^G$.

Mumford's geometric invariant theory(Mumford et al., 1994) establishes that under suitable conditions, there is an object Y, the "points" of which correspond to orbits in X and the "regular functions" of which form the ring $O_X(X)^G$. Furthermore, given any two orbits there is an element of $O_X(X)^G$ which distinguishes them.

The object Y is not an algebraic set, but rather a scheme, or in other words, a locally ringed space isomorphic to the spectrum of a commutative ring. Given an ideal I and corresponding algebraic set X, Mumford gives X a scheme structure by constructing its structure sheaf $\mathcal{O}_X$; the pair $(X, \mathcal{O}_X)$, and by abuse of notation X, shall be referred to as a *fusion scheme*. He then shows that under suitable conditions the spectrum of the ring $O_X(X)^G$, along with a suitable morphism of schemes, give the desired quotient structure.

All that said, however, the functions in $\mathcal{O}_X(X)^G$, i.e., the objects of our interest, are still classes of polynomial functions on X. For the reader who is willing to take the statements of a few theorems on faith, it is not necessary to understand anything about schemes, locally ringed spaces, or geometric invariant theory in order to appreciate the main result of this paper.

Lemma 5.2.3. *The orbits of G in X are closed.*

Proof. The action map $\cdot : G \times X \to X$ is a morphism of varieties. Then by (Humphreys, 1975, 8.3), in the Zariski topology one has for each orbit $G \cdot \mathbf{F}$ one has that

$$\overline{G \cdot \mathbf{F}} = G \cdot \mathbf{F} \bigcup_{\mathbf{E} \in (G \cdot \mathbf{F})'} G \cdot \mathbf{E}$$

and orbits of minimal dimension are closed. The stabilizer of $\mathbf{F}$ in G is independent of $\mathbf{F}$(Davidovich et al., 2013, Section 4.1), so all orbits are of the same dimension(Tauvel and Yu, 2005, 25.1.3), and thus closed. □

Corollary 5.2.4. *The orbits of G in X are irreducible.*

Proof. Follows from (Humphreys, 1975, 8.2, (d)), connectedness of G, and Ocneanu rigidity. □

Proposition 5.2.5. *Let X be a fusion scheme and G the associated gauge group acting as in Definition 5.2.1. Then there exists an algebraic scheme Y and a surjective morphism of schemes*

$$\psi : X \to Y$$

such that

1. $\mathcal{O}_Y(Y) = \mathcal{O}_X(X)^G$,
2. *for every point $y \in Y$, $\psi^{-1}(y)$ is an orbit and $\mathcal{O}_{Y,y} = \mathcal{O}_X(\psi^{-1}(y))^G$.*

Proof. The construction in (Mumford et al., 1994, Theorem 1.1) constructs

$$Y = Spec(\mathcal{O}_X(X)^G)$$

and ψ which satisfy (1) whenever G is a reductive group acting algebraically on X. In this case, G is a direct product of reductive groups, and thus reductive, and the action is clearly algebraic. In these circumstances, (Mumford et al., 1994, Amplification 1.3) asserts that (2) holds if and only if orbits are closed, which has already been shown. □

Remark 5.2.6. Since, in our case, Y is an algebraic scheme, up to isomorphism of schemes, it may be represented as an algebraic set. However, we wish to use information from the sheaf structure in Proposition 5.2.5 which is not isomorphism-invariant and without knowing the orbits beforehand there is no good way of choosing an isomorphism which preserves the sheaf structure on the nose.

We can now move on to proving the first half of Theorem 5.1.1:

Proposition 5.2.7. *Let $\mathscr{N} \geq 2$ be the number of orbits of G in X. Then there exist $\mathscr{N}-1$ G-invariant regular functions $f_1, \dots f_{\mathscr{N}-1}$ on X such that for all $\mathbf{F}, \mathbf{F}' \in X$ $\mathbf{F} \sim \mathbf{F}'$ if and only if*

$$f_i(\mathbf{F}) = f_i(\mathbf{F}')$$

for $i = 1, \dots, P$.

Proof. By (Mumford et al., 1994, Corollary 1.2), for any two distinct orbits $G \cdot \mathbf{F}$ and $G \cdot \mathbf{F}'$, there exists a G-invariant regular function $f : X \to k$ such that $f(G \cdot \mathbf{F}) = \{1\}$ and $f(G \cdot \mathbf{F}') = \{0\}$. Since the orbits are disjoint, f is the gluing of a function on $G \cdot \mathbf{F}$ with a function on $G \cdot \mathbf{F}'$. One then extends f to a function such that $f(G \cdot \mathbf{F}) = \{1\}$ and $f(G \cdot \mathbf{F}') = \{0\}$ on all orbits $G \cdot \mathbf{F}' \neq G \cdot \mathbf{F}$. We only need $\mathscr{N} - 1$ functions since for the $\mathscr{N}$-th orbit, $f_i(G \cdot \mathbf{F}_{\mathscr{N}}) = 0$ for all f_i and by construction this is the only orbit for which that is the case.

Now assume that $\mathbf{F}_p \sim \mathbf{F}_q$. The f_i are defined on all of X and are invariant under the G-action, so $f_i(\mathbf{F}_p) = f_i(\mathbf{F}_q)$ for all f_i. □

We now turn our attention to the action of $Aut(R, B)$ on X.

Definition 5.2.8. Define the action of $\rho \in Aut(R, B)$ on X via

$$(\rho \cdot F)^d_{abc} := F^{\rho^{-1}(d)}_{\rho^{-1}(a)\rho^{-1}(b)\rho^{-1}(c)}. \tag{5.8}$$

Since (4.15),(4.16), and (4.17) are symmetric under the action of ρ, $(\rho \cdot \mathbf{F}) \in X$. Call $\mathbf{F}, \mathbf{F}' \in X$ *permutation equivalent* if $(\rho \cdot \mathbf{F}) = \mathbf{F}'$ for some $\rho \in Aut(R, B)$.

For gauge equivalent $\mathbf{F}$ and $\mathbf{F}^g$, the triple $\mathcal{F} = (Id, Id, g)$ forms an object fixing monoidal functor $\mathcal{F} : \mathcal{C}(\mathbf{F}) \to \mathcal{C}(g \cdot \mathbf{F})$. $\mathcal{G}_\rho = (\rho, Id, Id)$ defines an object-permuting equivalence functor $\mathcal{G}_\rho : \mathcal{C}(\mathbf{F}) \to \mathcal{C}(\rho \cdot \mathbf{F})$. For any $\mathbf{F}, \mathbf{F}^{eq} \in X$ such that there exists a monoidal equivalence $\mathcal{H} : \mathcal{C}(\mathbf{F}) \to \mathcal{C}(\mathbf{F}^{eq})$, $\mathcal{H}$ factors into the composition of a permutation and gauge equivalence.

$Aut(R,B)$ acts on G by permuting labels and so we can define the action of $Aut(R,B) \ltimes G$ on X. It was shown in (Davidovich et al., 2013) that for a fusion category $\mathcal{C}$ associated to X, $Aut(R,B) \ltimes G \cong Aut^{\otimes}(\mathcal{C})$ – the group of monoidal auto-equivalences of $\mathcal{C}$. Y inherits an action by $Aut(R,B)$ and since Y has finitely many points, there are necessarily finitely many orbits of $Aut(R,B)$ in Y and they are necessarily closed. We can now prove the second half of Theorem 5.1.5:

Proposition 5.2.9. *Let X be a fusion scheme and Y be as in Proposition 5.2.5. Then there exists an algebraic scheme Z and a surjective morphism of schemes*

$$\eta : Y \to Z$$

such that

1. $\mathcal{O}_Z(Z) = \mathcal{O}_Y(Y)^{Aut(R,B)} = \mathcal{O}_X(X)^{Aut(R,B)\ltimes G}$,
2. *for every point z in Z, $\eta^{-1}(z)$ and $(\psi \circ \eta)^{-1}(z)$ are orbits in Y and X respectively and*

$$\mathcal{O}_{Z,z} = \mathcal{O}_Y(\eta^{-1}(z))^{Aut(R,B)} = \mathcal{O}_X((\psi \circ \eta)^{-1}(z))^{Aut(R,B)\ltimes G}.$$

Proof. Similar to Proposition 5.2.5 the construction

$$Z = Spec(\mathcal{O}_Y(Y))^{Aut(R,B)\ltimes G}$$

is used. The first statement is always true (Dolgachev, 2003, Example 6.1) and the second follows by composition of $\psi \circ \eta$ in the category of k-schemes. □

The second half of Theorem 5.1.1 then follows by the same logic as Proposition 5.2.7.

Corollary 5.2.10. *Let $\mathbf{F} \in X$ and $z = (\psi \circ \eta)(\mathbf{F})$ Then up to monoidal equivalence, the ring $\mathcal{O}_{Z,z}$ is a complete invariant of $\mathcal{C}(\mathbf{F})$.*

5.2.2 Invariants for Multiplicity Free Fusion Schemes

Assuming that (R, B) is multiplicity free, the stronger statements in 5.1.2 may be proved. Recall that multiplicity free means $N_{ab}^c \in \{0, 1\}$ for all $\gamma_{ab}^c \in \Gamma$. G is the set of functions $g : \Gamma \to k^\times$. Define $g_{ab}^c = g(\gamma_{ab}^c)$ and $g_c^{ab} = (g(\gamma_{ab}^c))^{-1}$. G acts on X via

$$(g \cdot F)_{abc}^d \left[\begin{smallmatrix} 1 & f \\ 1 & e \end{smallmatrix}\; 1\right] = (g_{ab}^e)(g_{ec}^d) F_{abc}^d \left[\begin{smallmatrix} 1 & f \\ 1 & e \end{smallmatrix}\; 1\right] (g_f^{bc})(g_d^{af}). \tag{5.9}$$

which is just what one obtains from (5.7) in the multiplicity free case. Note that since $g_{ab}^c \neq 0$, $(F^g)_{abc}^d \left[\begin{smallmatrix} 1 & f \\ 1 & e \end{smallmatrix}\; 1\right] = 0$ if and only if $F_{abc}^d \left[\begin{smallmatrix} 1 & f \\ 1 & e \end{smallmatrix}\; 1\right] = 0$. Thus, the set $F^{-1}(0)$ is a gauge invariant.

For any set S, define $A(S)$ to be the free Abelian group with basis S. For any $g \in G$, define $g_A : A(\Gamma) \to k^\times$ the extension of g to a group homomorphism. Rewriting (5.9) reveals that $\mathbf{F}$ and $\mathbf{F}'$ are equivalent if and only if there exists a function $g_A : A(\Gamma) \to k^\times$ such that for all $\{\gamma_{ab}^c, \gamma_{ec}^d, \gamma_{bc}^f, \gamma_{af}^d\} \subset \Gamma$,

$$g_A(\gamma_{ab}^e \gamma_{ec}^d (\gamma_{bc}^f \gamma_{af}^d)^{-1}) F_{abc}^d \left[\begin{smallmatrix} 1 & f \\ 1 & e \end{smallmatrix}\; 1\right] = (F')_{abc}^d \left[\begin{smallmatrix} 1 & f \\ 1 & e \end{smallmatrix}\; 1\right] \tag{5.10}$$

Define $t : \Phi \to A(\Gamma)$ such that

$$t(\Phi_{abc}^d \left[\begin{smallmatrix} 1 & f \\ 1 & e \end{smallmatrix}\; 1\right])) = \gamma_{ab}^e \gamma_{ec}^d (\gamma_{bc}^f \gamma_{af}^d)^{-1}$$

and $t_A : A(\Phi) \to A(\Gamma)$ its extension to to a group homomorphism. Then given a gauge transformation $g \in G$, the action of g on $\mathbf{F} \in X$ can then be rewritten as

$$(F^g)_{abc}^d \left[\begin{smallmatrix} 1 & f \\ 1 & e \end{smallmatrix}\; 1\right] = (t \circ g_A)(\Phi_{abc}^d \left[\begin{smallmatrix} 1 & f \\ 1 & e \end{smallmatrix}\; 1\right]) F_{abc}^d \left[\begin{smallmatrix} 1 & f \\ 1 & e \end{smallmatrix}\; 1\right] \tag{5.11}$$

Lemma 5.2.11. *Let $\mathbf{F}, \mathbf{F}' \in X$. Then $\mathbf{F} \sim \mathbf{F}'$ if and only if there exists a homomorphism $v : im(t_A) \to k^\times$ such that $\phi \in \Phi$,*

$$(t \circ v)(\phi) F(\phi) = F'(\phi) \tag{5.12}$$

Proof. Let g be a gauge transformation from $\mathbf{F}$ to $\mathbf{F}'$. Then g_A restricts to $im(t_A)$.

Conversely, suppose we have $v : im(t_a) \to k^\times$ satisfying equation (5.12) for every $\phi \in \Phi$. Since k is of characteristic 0, $k^\times$ is an injective $\mathbb{Z}$-module. Thus v extends to a well-defined homomorphism $\bar{v} : A(\Gamma) \to k^\times$, the set map restriction to Γ of which satisfies equations (5.10). □

Define $A_{Inv}(\Phi) := ker(t_A)$ and consider an element $A_{Inv}(\Phi) \ni m = \prod_{i=1}^{n}(\phi_i)^{k_i}$. Let $F(m) = \prod_{i=1}^{n} F(\phi_i)^{k_i}$ be the evaluation of m at $\mathbf{F} \in X$. If $F(m)$ is defined, then for all $g \in G$ the value $(g \cdot F)(m)$ is determined by g. By definition though, m is invariant under gauge transformation, and so $F(m) = (g \cdot F)(m)$. Thus if $F(m)$ is defined at $\mathbf{F}' \in G \cdot \mathbf{F}$, then $F(m)$ is defined at all points in $G \cdot \mathbf{F}$. Additionally, $F(m)$ is constant on $G \cdot \mathbf{F}$, and so there exists $f_m \in \mathcal{O}_X(G \cdot \mathbf{F})^G$ such that $f_m = F(m)$.

Proposition 5.2.12. *Let $\psi : X \to Y$ be a quotient map as in Proposition 5.2.5. Fix $\mathbf{F} \in X$ with $\psi(\mathbf{F}) = y \in Y$. The evaluations of monomials*

$$\begin{aligned} M_y &= \{m \in A_{Inv}(\Phi) | F(m) \text{ is well defined}\} \\ &= \{m | f_m \in \mathcal{O}_X(\psi^{-1}(y))^G\}. \end{aligned}$$

uniquely determine the evaluations of all $f_m \in \mathcal{O}_{Y,y}$.

Proof. M_y is a finitely generated multiplicative sub-monoid of $A_{Inv}(\Phi)$. By definition of $\mathcal{O}_{Y,y}$ there exists a monoid homomorphism

$$\zeta_M : M_y \to \mathcal{O}_{Y,y}$$

such that $\zeta(m) = f_m$.

Let $f_p \in \mathcal{O}_{Y,y}$. Locally, it is a ratio of polynomials $g = g_1 + \ldots + g_n$ and $h = h_1 + \ldots + h_m$ not identically zero such that $f_p = g/h$ on $\psi^{-1}(y)$. Without loss of generality, it may be assumed that neither g nor h contains variables ϕ_i such that $F(\phi_i) = 0$. It may also be

assumed that 1) g is monomial – since g/h can be broken into the (G-invariant) terms $g_1/h + \ldots + g_n/h$ – and 2) $F(g) \neq 0$.

This then implies that for all monomials $h_1, \ldots, h_m$ that $t_A(g) = t_A(h_j)$ for $j = 1, \ldots, m$. Thus we have that, on $\psi^{-1}(y)$,

$$f_p = \frac{g}{h} = \frac{1}{g^{-1}(h_1 + \ldots + h_m)} = \frac{1}{h_1(g^{-1}) + \ldots h_n(g^{-1})}.$$

By construction the terms $h_j(g^{-1})$ are nonzero on $\psi^{-1}(y)$ and thus $h_j(g^{-1}) \in M_y$. Thus f_p is representable by the inverse of a sum of terms in M_y. □

Fix $y \in Y$ and define the maximal subgroup $M_y \supset A_y = \{m \in M_y | \forall \mathbf{F} \in \psi^{-1}(y), F(m) \neq 0\}$. Since A_y is finitely generated, it has a basis $\mathcal{B}(A_y) = \{m_1, \ldots, m_n\}$. In Corollary 5.5.3 we show that A_y uniquely determines $F^{-1}(0)$ and thus M_y.

Corollary 5.2.13. *Fix a point $\mathbf{F} \in X$ such that $\psi(\mathbf{F}) = y$. The evaluations of monomials in $\mathcal{B}(A_y)$ uniquely determine the evaluations of all $f_m \in \mathcal{O}_{Y,y}$.*

Corollary 5.2.14. *Let $\mathbf{F}' \in X$ with $y' = \psi(\mathbf{F}')$, and $y \neq y'$. Then there exists a G-invariant monomial*

$$\prod_{i=1}^{n} \phi_i^{k_i} = m \in \mathcal{B}(A_y)$$

such that $F(m) \neq F'(m)$ or $F'(m)$ is undefined.

Corollary 5.2.15. *$\mathcal{C}(\mathbf{F})$ is gauge equivalent to its Galois conjugate if and only if $F(m) \in \mathbb{Q}$ for all $m \in \mathcal{B}(A_y)$.*

The first half of Theorem 5.1.3 has now been proved, as has the first half of Theorem 5.1.2. Before proving the second half of Theorem 5.1.2, we note that we have all of the information necessary to characterize the quotient Y.

Let $y_1, \ldots y_n \in Y$ be a collection of points with $A_{y_1}, \ldots, A_{y_n}$ respectively. Define the group

$$A_{y_1,\ldots,y_n} = \{m \in A_{inv}(\phi) | \forall \mathbf{F} \in \bigcup_{i=1}^{n} \psi^{-1}(y_i), F(m) \neq 0\}.$$

It is immediate that $A_{y_1,\ldots,y_n} = \bigcap_{i=1}^{n} A_{y_i}$ and so one make choices such that

$$\mathcal{B}(A_{y_1,\ldots,y_n}) = \bigcap_{i=1}^{n} \mathcal{B}(A_{y_i}).$$

Theorem 5.2.16. *Partition Y into set Y_k such that for all $y \in Y_j$, and $y' \in Y_k$, $A_y = A_{y'}$ if and only if $j = k$. The Y_k define closed sub-schemes which define a partition on Y.*

Proof. Y is zero dimensional and all points are closed so Y_k is clopen – hence it has the canonical structure of an open sub-scheme. Then for all $U \subset Y_k$, we define $\mathcal{O}_{Y_k}(U) := \mathcal{O}_{Y|Y_k}(U) = \mathcal{O}_Y(U)$. There is a canonical injection of Y_k into Y which is trivially a closed immersion, so Y_k is closed. Y is then the disjoint union of these schemes. □

Corollary 5.2.17. *Each $y \in Y_k$ is uniquely determined by the evaluations of monomials in $\mathcal{B}(A_{Y_k})$.*

In the multiplicity-free case, the action of $\rho \in Aut(R, B)$ on $\mathbf{F}$ as defined in (5.8) is equivalent to the action

$$\rho \cdot F^d_{abc} \begin{bmatrix} 1 & f & 1 \\ 1 & e & 1 \end{bmatrix} = F^{\rho(d)}_{\rho(a)\rho(b)\rho(c)} \begin{bmatrix} 1 & \rho(f) & 1 \\ 1 & \rho(e) & 1 \end{bmatrix}. \tag{5.13}$$

For all $\mathbf{F} \in X$, there is an induced an action of $Aut(R, B)$ on the orbit $G \cdot \mathbf{F}$ via $\rho \cdot (G \cdot \mathbf{F}) = G \cdot (\rho \cdot \mathbf{F})$ and hence an action of $Aut(R, B)$ on Y. Define the action of $Aut(R, B)$ on Φ via

$$\rho \cdot \Phi^d_{abc} \begin{bmatrix} 1 & f & 1 \\ 1 & e & 1 \end{bmatrix} = \Phi^{\rho(d)}_{\rho(a)\rho(b)\rho(c)} \begin{bmatrix} 1 & \rho(f) & 1 \\ 1 & \rho(e) & 1 \end{bmatrix}. \tag{5.14}$$

With this action, F is an $Aut(R, B)$-equivariant map.

Fix $y \in Y$. M_y extends linearly to a k-algebra R_y and ζ_M extends to a k-algebra homomorphism ζ_R. Define $R_{y_1,\ldots,y_n}$ in a similar fashion to $A_{y_1,\ldots,y_n}$.

Proposition 5.2.18. *Let $Aut(R, B)$ act on Φ as in (5.14). Then for all $y \in Y$ and $\rho \in Aut(R, B)$ there exists an isomorphism $\rho_R : R_y \to R_{\rho \cdot y}$.*

Proof. For $m = \prod_{i=1}^{n}(\phi_i)^{k_i} \in M_y$ one extends the action of (5.14) to $\rho \cdot m := \prod_{i=1}^{n}(\rho \cdot \phi_i)^{k_i} \in M_{\rho.y}$. □

Proposition 5.2.19. *$Aut(R, B)$ permutes the closed subschemes $Y_1, \ldots, Y_k$.*

Proof. Let Y' be a closed subscheme of Y as defined in 5.2.16. Then

$$R_{Y'} = \bigcap_{i=1}^{n} R_{y_i} = R_{y_1}.$$

$Aut(R, B)$ permutes points of Y, so $\rho \cdot y_i \neq \rho \cdot y_j$ if and only if $y_i \neq y_j$. The set $\rho \cdot Y' = \{\rho \cdot y_1, \ldots, \rho \cdot y_n\}$ is a closed sub-scheme for exactly the same reasons as in Theorem 5.2.16. If $A_{\rho \cdot y_1} \neq A_{y_1}$ then $Y' \cap (\rho \cdot Y') = \emptyset$. □

Corollary 5.2.20. *$\mathbf{F}, \mathbf{F}' \in X$ are monoidally equivalent if and only if there exists $\rho \in Aut(R, B)$ such that $A_{\rho \cdot y} = A_y$ and $F(m) = F'(\rho \cdot m)$, for all $m \in \mathcal{B}(A_y)$.*

Remark 5.2.21. Theorem 5.2.16 through Corollary 5.2.20 provide useful machinery for studying orbits. Given a set S of solutions to a set of pentagon equations, these may first be grouped according to the relation that for all $\mathbf{F}, \mathbf{F}' \in S$, $\mathbf{F} \sim \mathbf{F}$ if and only if there exists $\rho \in Aut(R, B)$ such that $F^{-1}(0) = \rho \cdot F'^{-1}(0)$. This guarantees that for $Y = \sqcup_{i=1}^{k} Y_i$ and ψ as in Proposition 5.2.5, $\mathbf{F} \sim \mathbf{F}'$ if and only if there exists Y_i such that $\psi(\mathbf{F}) \in Aut(R, B) \cdot Y_i$ and $\psi(\mathbf{F}') \in Aut(R, B) \cdot Y_i$.

From Proposition 5.2.19, one is free to choose a representative $F^{-1}(0)$. One determines the sets Y_k and $\eta(Y_k)$ using Proposition 5.2.13 and Corollary 5.2.20 where ρ need only be in the group

$$Stab(Y_k) = \{\rho \in Aut(R, B) | \forall \mathbf{F} \in \psi^{-1}(Y_k), \rho \cdot F^{-1}(0) = F^{-1}(0)\}.$$

The number of gauge equivalence classes in the set $Aut(R, B) \cdot Y_k$ is then $|Y_k||Aut(R, B) : Stab(Y_k)|$. An example of this is given in Section 5.4.2 in computing the gauge and monoidal classes for categories Grothendieck equivalent to $Rep(D(S_3))$.

Let Z and η be as in Proposition 5.2.9 and choose $z \in Z$. On $R_{\eta^{-1}(z)}$ the isomorphisms ρ_R (properly restricted) are in fact automorphisms, and one recovers an action of $Aut(R, B)$ on $R_{\eta^{-1}(z)}$.

Proposition 5.2.22. *Let $\eta : Y \to Z$ be a quotient map as in Proposition 5.2.9 and fix $z \in Z$. The evaluations of*

$$R_z = \{r \in R_{\eta^{-1}(z)} | \rho \cdot r = r, \forall \rho \in Aut(R, B)\}$$

uniquely determine the evaluations of all $f_r \in \mathcal{O}_{Z,z}$.

Proof. The proof of this is similar to the first part of the proof of Proposition 5.2.12. One starts with the map $\zeta_R : R_z \to \mathcal{O}_{Z,z}$ and notes that for any rational function g/h with $g, h \in R_{\eta^{-1}(z)}$ to be $Aut(R, B)$-invariant both g and h must be $Aut(R, B)$-invariant. □

Corollary 5.2.23. *Let $\mathbf{F}, \mathbf{F}' \in X$ such that $(\eta \circ \psi)(\mathbf{F}) = z$ and $(\eta \circ \psi)(\mathbf{F}') = z'$ and $z \neq z'$. Then there exists an $Aut(R, B)$-invariant linear combination of G-invariant monomials $l \in R_z$ such $F(l) \neq F'(l)$ or $F'(l)$ is undefined.*

With this the proof of 5.1.2 is finished.

Remark 5.2.24. It is important to note that the elements of R_z are not determined by the elements of $\{m \in M_{\eta^{-1}(z)} | \rho \cdot m = m, \forall \rho \in Aut(R, B)\}$. As a counter-example consider a fusion category Grothendieck equivalent to $\mathbb{Z}_3$ with $L = \{1, \omega, \omega^2\}$. The involution $\rho : L \to L, \rho(1) = 1, \rho(\omega) = \omega^2, \rho(\omega^2) = \omega$ is an automorphism of the fusion rules. The products

$$\Phi^1_{\omega\omega\omega} \left[\begin{smallmatrix} 1 & \omega^2 \\ 1 & \omega^2 \end{smallmatrix}\right] \Phi^{\omega}_{\omega\omega^2\omega} \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] \Phi^{\omega}_{\omega 1\omega} \left[\begin{smallmatrix} 1 & \omega \\ 1 & \omega \end{smallmatrix}\right] \qquad \textit{and}$$

$$\Phi^1_{\omega^2\omega^2\omega^2} \left[\begin{smallmatrix} 1 & \omega \\ 1 & \omega \end{smallmatrix}\right] \Phi^{\omega^2}_{\omega^2\omega\omega^2} \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right] \Phi^{\omega}_{\omega^2 1\omega^2} \left[\begin{smallmatrix} 1 & \omega^2 \\ 1 & \omega^2 \end{smallmatrix}\right]$$

are not invariant under ρ but their sum is.

5.3 Computing Invariants in Practice

In general, it's going to be difficult to determine the sets for Theorem 5.1.1 and Theorem 5.1.2. However, for the multiplicity-free case, this is surprisingly straightforward.

In Lemma 5.5.1 we prove that for $\mathbf{F}$ and $\mathbf{F}' \in X(R, B)$ with $\psi(\mathbf{F}) = y$ and $\psi(\mathbf{F}') = y'$,

$$F^{-1}(0) = (F')^{-1}(0) \Leftrightarrow M_y = M_{y'} \Leftrightarrow A_y = A_{y'}.$$

Then for $\mathbf{F} \in X(R, B)$ with $\psi(\mathbf{F}) = y \in Y(R, B)$ and $F^{-1}(0)$, Algorithm 1 computes a basis $\mathcal{B}(A_y)$ which from 5.2.13 is sufficient to check for gauge equivalence.

Algorithm 1 Computing a basis for gauge invariant products

procedure INVS($F^{-1}(0)$)
  $\Phi^\times \leftarrow \Phi - F^{-1}(0)$
  for $\phi_i \in \Phi^\times$ **do**
    $\mathbf{A}_{ij} \leftarrow (k_{ij} | t(\phi_i) = \prod_{j=1}^{|\Gamma|} \gamma_j^{k_{ij}})$ ▷ A_{ij} is the power of γ_j in $t(\phi_i)$.
  end for
  $(\mathbf{H}, \mathbf{HA}) \leftarrow hnf(\mathbf{A})$ ▷ Compute the Hermite Normal Form of $\mathbf{A}$
  $\mathcal{H} \leftarrow \{\mathbf{H}_i | (\mathbf{HA})_i$ is a zero vector.$\}$.
  for all $\mathbf{H}_i \in \mathcal{H}$ **do**
    $m_i \leftarrow \prod_{j=1}^{|\Phi^\times|} \phi_j^{\mathbf{H}_{ij}}, \phi_j \in \Phi^\times$ ▷ Compute the monomials.
  end for
end procedure
return $\mathcal{B}(A_y) = \{m_i | \mathbf{H}_i \in \mathcal{H}\}$. ▷ Return the basis $\mathcal{B}(A_y)$.

Again, for brevity fix X, Γ, and Φ. The image of $\phi \in \Phi$ under t_A is an element in $A(\Gamma)$, which can be represented as a vector of length $|\Gamma|$ with integer coefficients. This yields a $|\Gamma| \times |\Phi|$ matrix representing a system of $|\Phi|$ linear equations in $|\Gamma|$ variables which is equivalent to the equations (5.9).

Define $\Phi^\times = \Phi - F^{-1}(0)$ and the resulting $|\Gamma| \times |\Phi^\times|$ matrix to be $\mathbf{A}$. This system of equations is canonically solved by choosing orderings of Γ and $\Phi^\times$ then computing the Hermite normal form $\mathbf{HA}$. Computing $\mathbf{HA}$ is at worst an $\mathcal{O}(|\Phi||\Gamma|^4 \log^2(\mathcal{M}))$(Micciancio and Warinschi, 2001) computation where $\mathcal{M}$ is a bound on the entries of $\mathcal{A}$.

Given an ordering on $\Phi^\times$, the $|\Phi^\times| \times |\Phi^\times|$ matrix $\mathbf{H}$ is unique. Rows of $\mathbf{H}$ correspond to products of ϕ's whose gauge transformation is given by the same row of $\mathbf{HA}$. Thus the product in row i of $\mathbf{H}$ is gauge invariant if and only if row i of $\mathbf{HA}$ is identically zero. Those rows of $\mathbf{H}$ which correspond to identically zero rows of $\mathbf{HA}$ define a basis for A_y. This proves the second half of Theorem 5.1.3.

Remark 5.3.1. It is a natural question to ask whether or not the invariants from Theorem 5.1.2 can be obtained without solving the pentagon equations – i.e., whether or not one can compute Y directly. In Section 5.4.1 we provide a complete example from start to finish and discuss what may be necessary in general.

5.4 Examples for G-Invariants and Quotient Schemes

For simplicity, we will adopt the following notation for the rest of the section:

$$\Phi_{abc}^{d;ef} := \Phi_{abc}^{d} \left[\begin{smallmatrix} 1 & f \\ 1 & e \end{smallmatrix}\; 1\right]$$

In addition to the following examples, many more are computable from the Mathematica package listed in the introduction.

5.4.1 Quotient Scheme for Fibonacci Categories

As a first example which is short enough to compute explicitly, we re-derive the classification of fusion categories Grothendieck equivalent to representation category of the even half of $\mathfrak{su}(2)_3$. These are the Fibonacci and Yang-Lee categories. We will denote their Grothendieck ring by **Fib** with basis elements $\{1, \tau\}$ such with the non-trivial fusion rule $\tau \otimes \tau \cong 1 \oplus \tau$. The elements of $\Phi(\mathbf{Fib})$ and images under t_A are given in Table 5.1

The first step is to identify the sets $F^{-1}(0), \forall \mathbf{F} \in X$. It is well known that for all $\mathbf{F} \in X(\mathbf{Fib})$, $F_{abc}^{d;ef} \neq 0$ for all $\Phi_{abc}^{d;ef} \in \Phi(\mathbf{Fib})$. This can be determined by categorical considerations and by the procedure in (Bonderson, 2007, 2.5.2). There are no automorphisms,

and thus $Z(\mathbf{Fib}) = Y(\mathbf{Fib})$. Then, using the ordering in 5.1, the matrix $\mathbf{A}$ coming from equations of the form (5.9), $\mathbf{H}$, and $\mathbf{HA}$ are given in Figures 5.1 and 5.2. After reordering, these yield the basis for $A_{\mathbf{Fib}}$ given in Table 5.2.

Table 5.1. The variables Φ for Fibonacci and Yang-Lee categories.

$\Phi^{1;11}_{111}$	1	$\Phi^{\tau;1\tau}_{11\tau}$	$t^1_{11}(t^\tau_{1\tau})^{-1}$	$\Phi^{\tau;\tau\tau}_{1\tau1}$	1	$\Phi^{1;\tau1}_{1\tau\tau}$	$t^\tau_{1\tau}(t^1_{11})^{-1}$
$\Phi^{\tau;\tau\tau}_{1\tau\tau}$	1	$\Phi^{\tau;\tau1}_{\tau11}$	$t^\tau_{\tau1}(t^1_{11})^{-1}$	$\Phi^{1;\tau\tau}_{\tau1\tau}$	$t^\tau_{\tau1}(t^\tau_{1\tau})^{-1}$	$\Phi^{\tau;\tau\tau}_{\tau1\tau}$	$t^\tau_{\tau1}(t^\tau_{1\tau})^{-1}$
$\Phi^{1;1\tau}_{\tau\tau1}$	$t^1_{11}(t^\tau_{\tau1})^{-1}$	$\Phi^{\tau;\tau\tau}_{\tau\tau1}$	1	$\Phi^{1;\tau\tau}_{\tau\tau\tau}$	1	$\Phi^{\tau;11}_{\tau\tau\tau}$	$t^\tau_{1\tau}(t^\tau_{\tau1})^{-1}$
$\Phi^{\tau;1\tau}_{\tau\tau\tau}$	$t^1_{\tau\tau}t^\tau_{1\tau}(t^\tau_{\tau\tau})^{-2}$	$\Phi^{\tau;\tau1}_{\tau\tau\tau}$	$(t^\tau_{\tau\tau})^2(t^1_{\tau\tau}t^\tau_{\tau1})^{-1}$	$\Phi^{\tau;\tau\tau}_{\tau\tau\tau}$	1		

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 2 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{HA} = \begin{pmatrix} 1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 5.1. The matrices $\mathbf{A}$ and $\mathbf{HA}$ for $X(\mathbf{Fib})$.

Table 5.2. A basis for A_{fib}.

s_1	$\Phi^{1;11}_{111}$	s_2	$\Phi^{\tau;\tau\tau}_{1\tau1}$	s_3	$\Phi^{\tau;\tau\tau}_{1\tau\tau}$
s_4	$\Phi^{\tau;\tau\tau}_{\tau\tau1}$	s_5	$\Phi^{1;\tau\tau}_{\tau\tau\tau}$	s_6	$\Phi^{\tau;\tau\tau}_{\tau\tau\tau}$
s_7	$\Phi^{\tau;1\tau}_{11\tau}\Phi^{1;\tau1}_{1\tau\tau}$	s_8	$\Phi^{\tau;\tau1}_{\tau11}\Phi^{1;1\tau}_{\tau\tau1}$	s_9	$\Phi^{\tau;1\tau}_{11\tau}\Phi^{\tau;\tau1}_{\tau11}\Phi^{\tau;11}_{\tau\tau\tau}$
s_{10}	$\Phi^{1;\tau\tau}_{\tau1\tau}(\Phi^{\tau;1\tau}_{11\tau}\Phi^{\tau;\tau1}_{\tau11})^{-1}$	s_{11}	$\Phi^{\tau;\tau\tau}_{\tau1\tau}(\Phi^{\tau;1\tau}_{11\tau}\Phi^{\tau;\tau1}_{\tau11})^{-1}$	s_{12}	$\Phi^{\tau;1\tau}_{11\tau}\Phi^{\tau;\tau1}_{\tau11}\Phi^{\tau;1\tau}_{\tau\tau\tau}\Phi^{\tau;\tau1}_{\tau\tau\tau}$

Next, one divides out by non-zero terms in the pentagon equations to obtain a set of gauge invariant equations in the gauge invariant monomials as follows: For simplicity, let

$$\mathbf{H} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Figure 5.2. The matrix $\mathbf{H}$ for $X(\mathbf{Fib})$.

p_1 and p_2 be the only two nonzero terms in some (4.17) with $p_1 - p_2 = 0$. Without loss of generality, one may take $p_1(p_2)^{-1} = m_1 \in \mathcal{B}(A_y)$ and obtains

$$\begin{aligned}
(p_2)^{-1}(p_1 - p_2) &= 0 \\
\Rightarrow p_1(p_2)^{-1} - 1 &= 0 \\
\Rightarrow m_1 - 1 &= 0.
\end{aligned}$$

The last line is manifestly gauge invariant.

All equations of the form (4.17) can be localized in this way with respect to all non-zero terms. All terms of $Det(\Phi^d_{abc})$ have the same gauge class, and so can similarly be localized. There are only ever finitely many equations and finitely many terms and so there are only finitely many such localizations. This yields a finite set of equations in gauge invariant monomials, i.e., an ideal I in some $k[\mathcal{B}(A_{Y_i})]$.

Applying this procedure to $X(\mathbf{Fib})$ one finds that the evaluations for $s_1, \ldots, s_{12}$ are determined by the equations

$$s_1 = s_2 = s_3 = s_4 = s_5 = s_7 = s_8 = s_{10} = s_{11} = 1 \tag{5.15}$$

$$s_9 = s_{12} = -s_6 \tag{5.16}$$

$$(s_6)^2 - s_6 - 1 = 0. \tag{5.17}$$

The solutions to this are determined by $s_6 = \frac{1}{2}(1 \pm \sqrt{5})$. We note that the equations (5.15)-(5.17) are only those which determine the evaluations for all $s_i \in \mathcal{B}(A_{\mathbf{Fib}})$. We have omitted most of the relations defining the associated ideal I, but it is straightforward to compute that I is not radical. Thus Y is not itself an algebraic set.

While simple, this example encapsulates everything about the construction in Section 5.2. It's easy to see that one can completely reconstruct the associativity information for the categories from the evaluations of $s_1, \ldots, s_{12}$. In this way the result has another interpretation: $Y(\mathbf{Fib})$ is the moduli space for **Fib**. Its points are monoidal equivalence classes of categories and these are parameterized by the zero-loci of $x^2 - x - 1$.

Remark 5.4.1. From 3.4.2 it is obvious that the based ring for $D(\mathbf{Fib})$ is the product of two copies of the based ring for **Fib**. Particularly, this means that we have gauge invariant $6j$-symbols $F^{\tau;\tau\tau}_{\tau\tau\tau}$ and $F^{\bar{\tau};\bar{\tau}\bar{\tau}}_{\bar{\tau}\bar{\tau}\bar{\tau}}$. Both must satisfy the condition (5.17), and so these two are sufficient to determine the classification of $D(\mathbf{Fib})$ categories. Up to gauge equivalence, this gives us four possible theories, however there is an automorphism exchanging τ and $\bar{\tau}$ implying only three monoidal classes.

This example is made easier by the fact that for all $\mathbf{F} \in X(\mathbf{Fib}), F^{-1}(0) = \emptyset$. For categories without zeroes – e.g., the Tambara-Yamagami categories associated to a commutative group G – the above could be used to compute Y without modification.

In general, one must first identify the sets $F_i^{-1}(0)$ in order to compute a basis for the A_{Y_i}. For the case where $\mathcal{K}_0(\mathcal{C})$ has unitary categories, (Bonderson, 2007, 2.5.2) presents a heuristic algorithm which is conjectured to always determine the sets $F_i^{-1}(0)$ and thus the

groups A_{Y_i}. Considering the $F^{-1}(0)$ individually, one then exploits the partitioning Theorem 5.2.16.

5.4.2 Gauge and Monoidal Classes for $Rep(D(S_3))$

In this example we use the invariants of 5.2.2 to determine gauge and monoidal equivalence classes for fusion categories Grothendieck equivalent to the representation category of the quantum double of S_3. In this case, we take as our starting point a collection of solutions to the pentagon equations for $Rep(D(S_3))$. The number of matrices involved is large so rather than printing them they have been made available at http://mktitsworth.com/fusion-categories. This package includes the F and R matrices, pivotal coefficients, and a Mathematica notebook with the computations referenced here-in. Our classification is complete, however the solutions were found using computerized methods.

There are five monoidal equivalence classes – two for each set of i, j conditions on zeroes in the first row of Table 5.3 and one for the zeroes in the second row. All monoidal classes admit unitary structures – i.e., all F-matrices can be made unitary – but only four admit braidings. The allowable braidings correspond to the double and twisted double of S_3 as well as the two non-group theoretical categories discovered by (Gelaki et al., 2009). The modular data for these categories is readily available there.

To our knowledge the final equivalence class is new. In addition to those properties below which distinguish it from all others, its pivotal structures are also distinct: In the braided cases, the Frobenius-Schur indicators for the $\alpha_\pm$ objects are always equal, and in the non-braided case they are opposite. It is straightforward to compute this given the F-matrices.

Classification by Invariants

We proceed via our observations 5.2.16 and 5.2.19. We first partition the points of $Y(Rep(D(S_3)))$ via the equivalence relation

$$y_i \sim y_j \Leftrightarrow M_y = M_{y'} \Leftrightarrow F^{-1}(0) = F'^{-1}(0), \forall \mathbf{F} \in \psi^{-1}(y) \text{ and } \forall F' \in \psi^{-1}(y).$$

We then partition the points of $Z(Rep(D(S_3)))$ via

$$z_i \sim z_j \Leftrightarrow \forall y_k \in \eta^{-1}(z_i) \exists y_l \in \eta^{-1}(z_j) \text{ and } \rho \in Aut(D(S_3)) | M_{y_k} = M_{\rho \cdot y_l}$$

which is to say that the closed sub-schemes as defined in 5.2.16 for y_k is permuted by $\rho \in Aut(R, B)$ to that of y_l. We then determine a set of nonzero invariants $m_1, \ldots, m_8$ defined on all of $Y(Rep(D(S_3)))$, each of which distinguishes between gauge equivalence classes and whose sum determines monoidal classes.

Table 5.3. Table of zero sets for solutions to $Rep(D(S_3))$ pentagon equations.

$\Phi^{\alpha_\pm;\alpha_\pm\alpha_\mp}_{\beta_i\alpha_\pm\beta_j}$, $\Phi^{\alpha_\pm;\alpha_\mp\alpha_\pm}_{\beta_i\alpha_\pm\beta_j}$, $\Phi^{\alpha_\mp;\alpha_\pm\alpha_\pm}_{\beta_i\alpha_\pm\beta_j}$, $\Phi^{\alpha_\mp;\alpha_\mp\alpha_\mp}_{\beta_i\alpha_\pm\beta_j}$ $\Phi^{\beta_j;\alpha_\pm\alpha_\mp}_{\alpha_\pm\beta_i\alpha_\pm}$, $\Phi^{\beta_i;\alpha_\pm\alpha_\mp}_{\alpha_\pm\beta_i\alpha_\pm}$, $\Phi^{\beta_j;\alpha_\pm\alpha_\pm}_{\alpha_\pm\beta_i\alpha_\mp}$, $\Phi^{\beta_j;\alpha_\mp\alpha_\mp}_{\alpha_\pm\beta_i\alpha_\mp}$ $\Phi^{\alpha_\mp;\beta_i\beta_j}_{\alpha_\pm\alpha_\pm\alpha_\pm}$, $\Phi^{\alpha_\pm;\beta_i\beta_j}_{\alpha_\pm\alpha_\pm\alpha_\mp}$, $\Phi^{\alpha_\pm;\beta_i\beta_j}_{\alpha_\pm\alpha_\mp\alpha_\pm}$, $\Phi^{\alpha_\pm;\beta_i\beta_j}_{\alpha_\mp\alpha_\pm\alpha_\pm}$	$i \neq j \in \{\beta_1, \beta_2\}$ or $i = j \in \{\beta_3, \beta_4\}$ or $i \neq j \in \{\beta_1, \beta_2\}$ or $i \neq j \in \{\beta_3, \beta_4\}$
$\Phi^{\alpha_\pm;\alpha_\pm\alpha_\mp}_{\beta_i\alpha_\pm\beta_i}$, $\Phi^{\alpha_\pm;\alpha_\mp\alpha_\pm}_{\beta_i\alpha_\pm\beta_i}$, $\Phi^{\alpha_\mp;\alpha_\pm\alpha_\pm}_{\beta_i\alpha_\pm\beta_i}$, $\Phi^{\alpha_\mp;\alpha_\mp\alpha_\mp}_{\beta_i\alpha_\pm\beta_i}$ $\Phi^{\beta_i;\alpha_\pm\alpha_\mp}_{\alpha_\pm\beta_i\alpha_\pm}$, $\Phi^{\beta_i;\alpha_\mp\alpha_\pm}_{\alpha_\pm\beta_i\alpha_\pm}$, $\Phi^{\beta_i;\alpha_\pm\alpha_\pm}_{\alpha_\pm\beta_i\alpha_\mp}$, $\Phi^{\beta_i;\alpha_\mp\alpha_\mp}_{\alpha_\pm\beta_i\alpha_\mp}$ $\Phi^{\alpha_\mp;\beta_i\beta_i}_{\alpha_\pm\alpha_\pm\alpha_\pm}$, $\Phi^{\alpha_\pm;\beta_i\beta_i}_{\alpha_\pm\alpha_\pm\alpha_\mp}$, $\Phi^{\alpha_\pm;\beta_i\beta_i}_{\alpha_\pm\alpha_\mp\alpha_\pm}$, $\Phi^{\alpha_\pm;\beta_i\beta_i}_{\alpha_\mp\alpha_\pm\alpha_\pm}$	

Given that the solutions to the pentagon equations for $Rep(D(S_3))$ are taken as input, the zero sets are readily available. The allowable zeroes are also constructable by using

(Bonderson, 2007, 2.5.2) iteratively – determining the zeroes for the $Rep(S_3)$ subcategories, for $Rep(D(S_3))_{ad}$, and finally $Rep(D(S_3))$.

For the $Rep(S_3)$ subcategories the only zero is $\Phi^{\beta_i;\beta_i\beta_i}_{\beta_i,\beta_i,\beta_i}$. For $Rep(D(S_3))_{ad}$, all entries of the form $\Phi^{\beta_j;\beta_k\beta_k}_{\beta_i\beta_j\beta_i}$ are zero in all solutions and this set is invariant under permutation. For $Rep(D(S_3))$, up to permutation there are three sets of zeroes all sharing those of $Rep(D(S_3))_{ad}$ with additional zeroes listed in Table 5.3. Two sets of zeroes share form given in the first row/column of Table 5.3. They are distinguished by the conditions on the two dimensional objects which appear given in the first row/second column.

By inspection of Table 5.3 the first zero set is fixed by the Klein 4-group generated by $\langle(\beta_1,\beta_2),(\beta_3,\beta_4)\rangle$ plus (α_+,α_-). This is a subgroup of order eight and so there are six orbits – i.e., we can obtain six sets of zeroes from those in Table 5.3. The second set is fixed by the dihedral group D_4 generated by $\langle(\beta_1,\beta_2),(\beta_1,\beta_3,\beta_2,\beta_4)\rangle$ plus (α_+,α_-). This is a subgroup of order sixteen so there are three orbits. The final set of zeros in Table 5.3 is invariant under the action of $Aut(D(S_3))$. Thus we have ten total zero sets and ten subschemes $Y_1,\ldots,Y_{10}$.

At this point the classification is easy. The invariant products

$$\Phi^{\alpha_\pm;\beta_i\beta_i}_{\alpha_\pm\alpha_\pm\alpha_\pm}\Phi^{\beta_i;\alpha_\pm\alpha_\pm}_{\alpha_\pm\beta_i\alpha_\pm}$$

are defined on all of Y and each takes two values on each Y_i. Their values are given in Table 5.4. Thus there are 20 gauge classes. Their sum

$$\sum_{i=1}^{4}\Phi^{\alpha_+;\beta_i\beta_i}_{\alpha_+\alpha_+\alpha_+}\Phi^{\beta_i;\alpha_+\alpha_+}_{\alpha_+\beta_i\alpha_+}+\sum_{i=1}^{4}\Phi^{\alpha_-;\beta_i\beta_i}_{\alpha_-\alpha_-\alpha_-}\Phi^{\beta_i;\alpha_-\alpha_-}_{\alpha_-\beta_i\alpha_-}$$

is invariant under the action of $Aut(D(S_3))$ and takes five values, so there are five monoidal classes.

Table 5.4. Nonzero invariants which determine the gauge and monoidal equivalence classes.

Solution set, permutation	$Inv(D(S_3)) = \{\Phi^{\alpha_\pm;\beta_i\beta_i}_{\alpha_\pm\alpha_\pm\alpha_\pm}\Phi^{\beta_i;\alpha_\pm\alpha_\pm}_{\alpha_\pm\beta_i\alpha_\pm} \mid i=1,2,3,4\}$	$\sum_{i\in Inv(D(S_3))} i$
Zero set 1, unpermuted	$\{\pm\frac{2}{3},\pm\frac{2}{3},\pm\frac{1}{6},\pm\frac{1}{6},\pm\frac{2}{3},\pm\frac{2}{3},\pm\frac{1}{6},\pm\frac{1}{6}\}$	$\pm\frac{10}{3}$
Zero set 1, (β_2,β_3)	$\{\pm\frac{2}{3},\pm\frac{1}{6},\pm\frac{2}{3},\pm\frac{1}{6},\pm\frac{2}{3},\pm\frac{1}{6},\pm\frac{2}{3},\pm\frac{1}{6}\}$	$\pm\frac{10}{3}$
Zero set 1, $(\beta_2,\beta_3,\beta_4)$	$\{\pm\frac{2}{3},\pm\frac{1}{6},\pm\frac{1}{6},\pm\frac{2}{3},\pm\frac{2}{3},\pm\frac{1}{6},\pm\frac{1}{6},\pm\frac{2}{3}\}$	$\pm\frac{10}{3}$
Zero set 1, $(\beta_1,\beta_3,\beta_2)$	$\{\pm\frac{1}{6},\pm\frac{2}{3},\pm\frac{2}{3},\pm\frac{1}{6},\pm\frac{1}{6},\pm\frac{2}{3},\pm\frac{2}{3},\pm\frac{1}{6}\}$	$\pm\frac{10}{3}$
Zero set 1, $(\beta_1,\beta_3,\beta_4,\beta_2)$	$\{\pm\frac{1}{6},\pm\frac{2}{3},\pm\frac{1}{6},\pm\frac{2}{3},\pm\frac{1}{6},\pm\frac{2}{3},\pm\frac{1}{6},\pm\frac{2}{3}\}$	$\pm\frac{10}{3}$
Zero set 1, $(\beta_1,\beta_3),(\beta_2,\beta_4)$	$\{\pm\frac{1}{6},\pm\frac{1}{6},\pm\frac{2}{3},\pm\frac{2}{3},\pm\frac{1}{6},\pm\frac{1}{6},\pm\frac{2}{3},\pm\frac{2}{3}\}$	$\pm\frac{10}{3}$
Zero set 2, all	$\{\pm\frac{1}{6},\pm\frac{1}{6},\pm\frac{1}{6},\pm\frac{1}{6},\pm\frac{1}{6},\pm\frac{1}{6},\pm\frac{1}{6},\pm\frac{1}{6}\}$	$\pm\frac{4}{3}$
Zero set 3, unpermuted	$\{\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},-\frac{2}{3},-\frac{2}{3},-\frac{2}{3},-\frac{2}{3}\}$	0
Zero set 3, (α_+,α_-)	$\{-\frac{2}{3},-\frac{2}{3},-\frac{2}{3},-\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3},\frac{2}{3}\}$	0

5.5 Nonzero Invariant Monomials Determine All Invariant Monomials

For the entirety of this section let X be a fusion scheme with $\mathbf{F}, \mathbf{F}' \in X$ and Y a quotient scheme as in Proposition 5.2.12 with $\psi(\mathbf{F}) = y$ and $\psi(\mathbf{F}') = y'$. The point of this appendix is to prove the following:

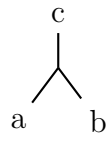
Lemma 5.5.1. *The following are equivalent:*

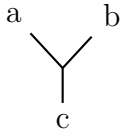
1. $F^{-1}(0) = (F')^{-1}(0)$,
2. $M_y = M_{y'}$, *and*
3. $A_y = A_{y'}$.

Proof. The direction $1 \Rightarrow 2 \Rightarrow 3$ is by construction. $3 \Rightarrow 1$ is a corollary of Lemma 5.5.2 and Corollary 5.5.3. □

The proof of Lemma 5.5.2 is made easier by using the graphical calculus for fusion categories, so for completeness, we provide a quick review. Since our proof is for multiplicity free categories, we will omit indices at vertices. For a full treatment of the graphical calculus see (Davidovich et al., 2013).

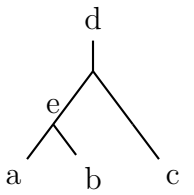
Let $\mathcal{C}$ be a fusion category. Basis vectors for hom spaces $\mathcal{C}(a \otimes b, c)$ may be represented graphically as

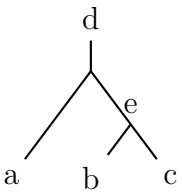


$\in \mathcal{C}(a \otimes b, c)$ $\qquad$ 

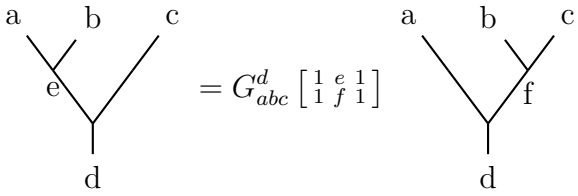
$\in \mathcal{C}(c, a \otimes b)$

Additionally we will always choose the basis for $\mathcal{C}(c, a \otimes b)$ which is algebraically dual to the basis for $\mathcal{C}(a \otimes b, c)$. Graphically, the numbers $F^d_{abc}\left[\begin{smallmatrix} 1 & f & 1 \\ 1 & e & 1 \end{smallmatrix}\right]$ are coefficients in the expression



$= F^d_{abc}\left[\begin{smallmatrix} 1 & f & 1 \\ 1 & e & 1 \end{smallmatrix}\right]$ 

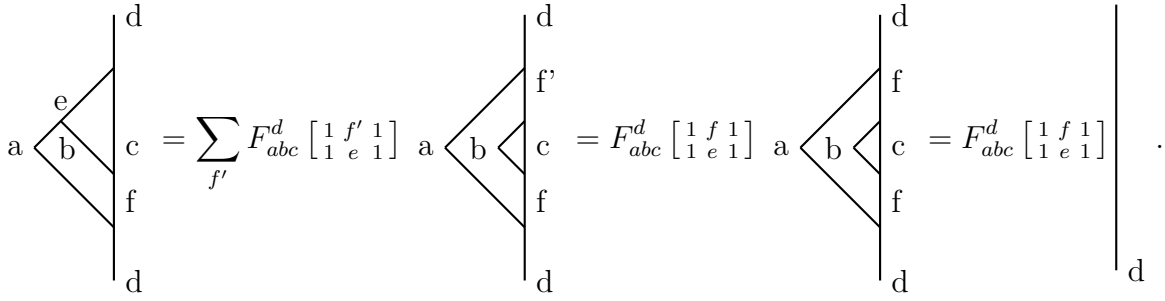
which relates the two different ways in which bases for $\mathcal{C}(a\otimes b\otimes c, d)$ spaces can be decomposed given bases for the spaces $\mathcal{C}(a\otimes b, c)$. One writes a similar expression for the splitting spaces



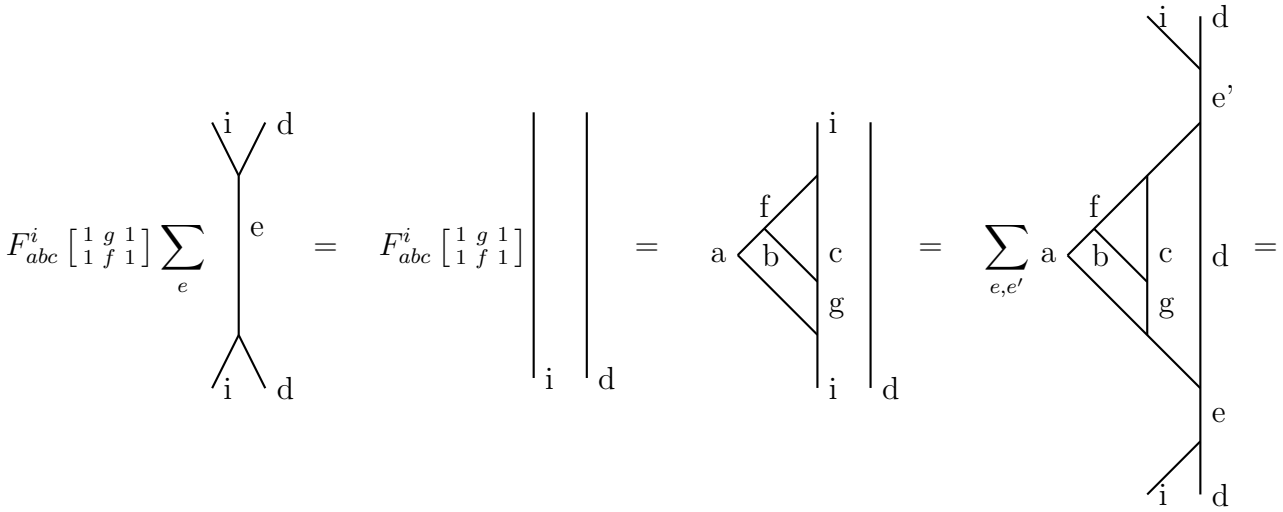
where G^d_{abc} is the inverse matrix to F^d_{abc} satisfying equivalent relations to (4.16) and (4.17).

Lemma 5.5.2. *Every $\phi \in \Phi$ such that $F(\phi) \neq 0$ has non-zero power in some $m \in A_y$.*

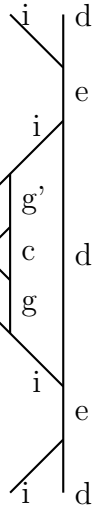
Proof. The case where $m = \phi$ (i.e., ϕ is gauge invariant) is trivial. Recall that $F(\phi) = F^d_{abc}\left[\begin{smallmatrix}1 & f & 1\\ 1 & e & 1\end{smallmatrix}\right]$ for some $\{a, b, c, d, e, f\}$ and consider the morphism given by



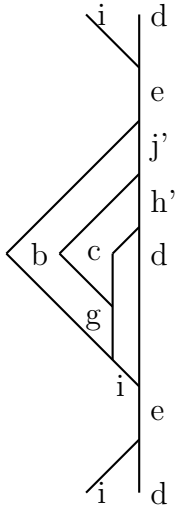
We then fix an object i and tensor this with some d to obtain



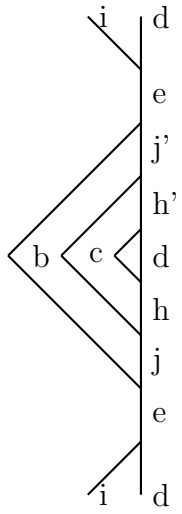
$$\sum_{e,g'} F^i_{abc}\left[\begin{smallmatrix}1 & g' & 1\\ 1 & f & 1\end{smallmatrix}\right]$$



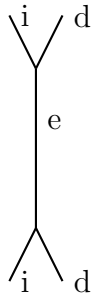
$$= \sum_{e,h',j'} F^i_{abc}\left[\begin{smallmatrix}1 & g & 1\\ 1 & f & 1\end{smallmatrix}\right] F^e_{agd}\left[\begin{smallmatrix}1 & j & 1\\ 1 & i & 1\end{smallmatrix}\right] F^j_{bcd}\left[\begin{smallmatrix}1 & h & 1\\ 1 & g & 1\end{smallmatrix}\right]$$



$$= \sum_{e,h,h',j,j'} F^i_{abc}\left[\begin{smallmatrix}1 & g & 1\\ 1 & f & 1\end{smallmatrix}\right] F^e_{agd}\left[\begin{smallmatrix}1 & j & 1\\ 1 & i & 1\end{smallmatrix}\right] F^j_{bcd}\left[\begin{smallmatrix}1 & h & 1\\ 1 & g & 1\end{smallmatrix}\right] G^e_{agd}\left[\begin{smallmatrix}1 & i & 1\\ 1 & j & 1\end{smallmatrix}\right] G^j_{bcd}\left[\begin{smallmatrix}1 & g & 1\\ 1 & h & 1\end{smallmatrix}\right]$$



$$= F^i_{abc}\left[\begin{smallmatrix}1 & g & 1\\ 1 & f & 1\end{smallmatrix}\right] \left(\sum_{e,h,j} F^e_{agd}\left[\begin{smallmatrix}1 & j & 1\\ 1 & i & 1\end{smallmatrix}\right] F^j_{bcd}\left[\begin{smallmatrix}1 & h & 1\\ 1 & g & 1\end{smallmatrix}\right] G^e_{agd}\left[\begin{smallmatrix}1 & i & 1\\ 1 & j & 1\end{smallmatrix}\right] G^j_{bcd}\left[\begin{smallmatrix}1 & g & 1\\ 1 & h & 1\end{smallmatrix}\right]\right)$$



(5.18)

This morphism is a multiple $F^i_{abc}\left[\begin{smallmatrix}1 & g & 1\\ 1 & f & 1\end{smallmatrix}\right]$ of the identity on $i \otimes d$, and we have that at least one of the terms

$$F^i_{abc}\left[\begin{smallmatrix}1 & g & 1\\ 1 & f & 1\end{smallmatrix}\right] F^e_{agd}\left[\begin{smallmatrix}1 & j & 1\\ 1 & i & 1\end{smallmatrix}\right] F^j_{bcd}\left[\begin{smallmatrix}1 & h & 1\\ 1 & g & 1\end{smallmatrix}\right]$$

must be non-zero. Each of these is an evaluation of a cubic term in some (4.17) and every equation of the form (4.17) has either zero or at least two nonzero terms. In the multiplicity free case, given an equation of the form (4.17) all terms in the equation transform as $t^f_{ab}t^i_{fc}t^e_{id}(t^e_{aj}t^j_{bh}t^h_{cd})^{-1}$. The ratio of any two terms is in $A_{Inv}(\Phi)$ and the ratio of any two non-zero terms is in A_y. □

Corollary 5.5.3. *A_y uniquely determines $F^{-1}(0)$.*

Lemma 5.5.1 is then a corollary of Lemma 5.5.2.

CHAPTER 6

CLASSIFICATION OF METAPLECTIC FUSION CATEGORIES

In this chapter we will showcase the power of the techniques presented in the previous chapter by applying them to the infinite family of fusion categories underlying metaplectic anyonic systems. Metaplectic anyons are called this because they give rise to metaplectic link invariants(Goldschmidt and Jones, 1989; Hastings et al., 2014, 2013). By themselves these are not universal for quantum computing however (Cui and Wang, 2015) provide a way to supplement braiding with measurement to obtain universality.

A related result on the classification of metaplectic *modular* categories was also recently obtained by (Ardonne et al., 2016). The classification of fusion categories contained in here is separate, since there are instances where multiple modular categories arise from the same fusion category.

The work in this chapter comes from the manuscript in final preparation by the author, Eddy Ardonne, and Peter Finch. The author was approached to help determine the equivalence classes for a collection of solutions to the pentagon and hexagon equations produced by Eddy, presented in 6.2.2 and 6.2.3. The author's primary contribution to this work is the determination of gauge and monoidal equivalence using the invariants of Chapter 5 in Sections 6.3.1 and 6.3.3. The work in 6.3.2 is the result of Peter Finch. The author is also responsible for 6.1 and working out the examples in 6.4, the modular data in 6.2.4, the notation of section 6.2.1 and proof of proposition 6.2.1, the notation in section 6.2.2, and the proof of proposition 6.2.2.

6.1 Introduction

In this chapter we obtain answer these questions for a family of categories we will call metaplectic fusion categories. These are fusion categories underlying metaplectic modular categories. Recall that a metaplectic modular category(Hastings et al., 2014, 2013) is a modular category Grothendieck equivalent to $\mathfrak{so}(2p+1)_2$, the category of affine $\mathfrak{so}(2p+1)$ representations with highest integer weight 2(Gannon, 2002). Other common constructions for such categories come from $\mathcal{C}(B_p, 2(2p+1), q)$(Rowell, 2006) for q some $2(2p+1)$ root of unity, or from $\mathbb{Z}_2$ equivariantizations of $\mathcal{TY}(\mathbb{Z}_N, \chi, \tau)$(Gelaki et al., 2009) where $N = 2p+1$. In (Ardonne et al., 2016, Theorem 3.2) it was shown that all metaplectic modular categories arise from gauging the particle-hole symmetry in $\mathbb{Z}_{2p+1}$ modular categories.

Our main result is obtained by classifying solutions to the pentagon equations and hexagon equations coming from the $\mathfrak{so}(2p+1)_2$ based ring. We stop short of demonstrating that these are all of the categories as this is a much harder problem. However, the modular data for $\mathfrak{so}(2p+1)_2$ categories is known and the modular data computed for our solutions coincides. As such, we refine the classification of (Ardonne et al., 2016) one step further.

For fixed p, our categories are parameterized by pairs (r, κ) where $\kappa = \pm 1$ and r a positive odd integer less than and co-prime to $2p+1$. Let R be the set of these r. Define $\mathbb{G}^{\times}_{2p+1} = \mathbb{Z}^{\times}_{2p+1}/\langle 1, -1\rangle$ with $g : \mathbb{Z}^{\times}_{2p+1} \to \mathbb{G}^{\times}_{2p+1}$ the quotient map. Elements of $\mathbb{Z}^{\times}_{2p+1}$ can be represented as positive integers less than and co-prime to $2p+1$. Likewise, elements of $\mathbb{G}^{\times}_{2p+1}$ can be represented as positive integers less than or equal to p and co-prime to $2p+1$. Particularly, each $r \in R$ represents an element of $\mathbb{Z}^{\times}_{2p+1}$ and the action of $\mathbb{G}^{\times}_{2p+1}$ on R via $z \cdot R = \{rz^2 \mod (2p+1) | r \in R\}$ makes sense. Similarly, the evaluation $g(r)|r \in R$ makes sense. Our main result can then be stated as follows

Theorem 6.1.1. *For fixed p the following are true:*

1. *The monoidal classes of fusion categories constructed from (r, κ), $\kappa = \pm 1$ and $r \in R$ are the fusion categories underlying metaplectic modular categories.*

2. *Let (r, κ) and (r', κ') parameterize two different solutions to the pentagon equations. Then the fusion categories constructed from these solutions are monoidally equivalent if and only if $\kappa = \kappa'$ and there exists $z \in \mathbb{G}_{2p+1}^{\times}$ such that $g(r') = g(rz^2)$.*

3. *For $2p + 1 = p_1^{a_1} \dots p_l^{a_l}$, there are exactly 2^{l+1} monoidally inequivalent metaplectic modular categories if $\exists b \in Z_{2p+1}^{\times} | b^2 = -1$, otherwise there are exactly 2^l.*

As a sanity check on this, following (Ardonne et al., 2016, Theorem 1.1), we note that for fixed p the number of metaplectic fusion categories is always less than or equal to the number of metaplectic modular categories, which is 2^{l+1}.

The structure of this chapter is as follows. In 6.2 we present our solutions to the pentagon and hexagon equations, and construct the modular data. In 6.1 we classify these solutions using the invariants of 5.2.2. In 6.4 we explicitly compute the equivalence classes and modular structures for several examples.

6.2 $\mathfrak{so}(2p+1)_2$ Fusion Systems

In the following we present arithmetic data for the F and R matrices which makes much of the structure for our categories apparent. We also provide explicit computations of the relevant categorical quantities.

There are several contexts in which the Grothendieck rings for our categories naturally arise, though two primary sources both come from the study of Lie algebras. As should be apparent from our notation, one of these is as Grothendieck rings for the categories of representations of the (untwisted) affine Kac-Moody algebras $B_p^{(1)}$ (i.e., $\mathfrak{so}(2p+1)$) with highest integral weight 2. As determined by (Finkelberg, 1996), these are equivalent (as modular categories) to the semi-simplification (a la (Andersen, 1992)) of the representation category for $U_q(B_p)$ with $q = \mathbf{e}^{\frac{\pi\imath}{2p+1}}$. The affine Kac-Moody and quantum group constructions for the Grothendieck rings can be found in (Gannon, 2002) and (Rowell, 2006) respectively.

Another source for our fusion rules comes from Tambara-Yamagami categories as follows. Let p a positive integer, χ a symmetric bi-character on $\mathbb{Z}_{2p+1}$ and $\nu = \pm$. In (Gelaki et al., 2009) it was shown that $\mathbb{Z}_2$-de-equivariantizations of the Tambara-Yamagami category $\mathcal{TY}(\mathbb{Z}_{2p+1}, \chi, \nu)$ gives rise to an $\mathfrak{so}(2p+1)_2$ category.

6.2.1 Fusion Rules for $\mathfrak{so}(2p+1)_2$ Categories

For some $\mathfrak{so}(2p+1)_2$ category $\mathcal{C}$, we label the basis elements of $K_0(\mathcal{C})$ (i.e., equivalence classes of simple objects) by elements of the set $L = \{\mathbf{1}, \epsilon, \phi_i, \psi_\pm\}$ with $i = 1, \dots p$. These have Frobenius-Perron dimensions $\{1, 1, 2, \sqrt{2p+1}\}$ respectively. $K_0(\mathcal{C})$ is commutative with non-trivial products given by:

$$\epsilon \otimes \epsilon \cong \mathbf{1} \qquad \phi_i \otimes \phi_i \cong \mathbf{1} \oplus \epsilon \oplus \phi_{g(2i)} \qquad \psi_\pm \otimes \psi_\pm \cong \mathbf{1} \bigoplus_{j=1}^{p} \phi_j$$

$$\epsilon \otimes \phi_i \cong \phi_i \qquad \phi_i \otimes \phi_j \cong \phi_{g(i-j)} \oplus \phi_{g(i+j)} \qquad \psi_\pm \otimes \psi_\mp \cong \epsilon \bigoplus_{j=1}^{p} \phi_j$$

$$\epsilon \otimes \psi_\pm \cong \psi_\mp \qquad \phi_i \otimes \psi_\pm \cong \psi_\pm \otimes \psi_\mp$$

where $g : \mathbb{Z}_{2p+1} \to \{0, \dots p\}$ is given by

$$g(a) = |a|.$$

Proposition 6.2.1. *Let $K_0(\mathcal{C})$ be the Grothendieck ring of a $\mathfrak{so}(2p+1)_2$ category. Then*

1. *the automorphisms which permute the ϕ_i are given by $\mathbb{G}^\times_{2p+1} := \mathbb{Z}^\times_{2p+1}/\langle 1, -1\rangle$,*
2. *the automorphisms which permute the $\psi_\pm$ are given by $\mathbb{Z}_2$,*
3. *and the automorphism group of $K_0(\mathcal{C})$ is $\mathbb{G}^\times_{2p+1} \times \mathbb{Z}_2$.*

Proof. That the fusion rules are invariant under exchange of $\psi_\pm$ is straight forward.

The automorphisms which permute the ϕ_i follow restricting g to $\mathbb{Z}_{2p+1}^\times$ and promoting it to a group homomorphism. For ease, consider the presentation of $\mathbb{Z}_{2p+1}$ as integers in $\{-p,\ldots,0,\ldots,p\}$. $\mathbb{Z}_{2p+1}^\times$ has even order [1] and can be represented by integers in $\pm 1,\ldots,\pm p$. If i is in $\mathbb{Z}_{2p+1}^\times$ so is $-i$, thus the quotient $\mathbb{Z}_{2p+1}^\times/\langle 1,-1\rangle$ is well defined. g is precisely this quotient map.[2] $\mathbb{G}_{2p+1}^\times$ acts on ϕ_i in the obvious way and since g is a group homomorphism, this preserves the fusion rules. There are no automorphisms on pointed objects or between objects of different Frobenius-Perron dimension, and so the automorphism group of $K_0(\mathcal{C})$ must be the direct product. This gives a separate proof from that in (Gannon, 2002). □

6.2.2 F-Matrices

Our solutions to (4.15)-(4.17) are indexed by pairs (r,κ) where r is an odd integer between 1 and $2p+1$, such that $gcd(r,2p+1)=1$ and $\kappa=\pm 1$. For a given r, the solution with $\kappa=1$ has all Frobenius-Schur indicators positive. For $\kappa=-1$ the second Frobenius-Schur indicator of the $\sqrt{2p+1}$ objects becomes -1.

Notation

Fix $q=\mathbf{e}^{\frac{\pi\imath}{2p+1}}$. To write the general F-symbols, we first introduce the following matrices.

$$A(s_1,s_2,s_3,s_4)=\frac{1}{\sqrt{2}}\begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} \qquad B=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad C=\frac{1}{2}\begin{pmatrix} 1 & -1 & \sqrt{2} \\ -1 & 1 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{pmatrix}$$

[1] Specifically the order has to be $\varphi(2p+1)$ where φ is Euler's totient function. For power k of prime r $\varphi(r^k)=r^{(k-1)}(r-1)$, which is even. That the evaluation of φ is even for every $2p+1$ follows from prime factorization and the multiplicative property of φ.

[2] There is not an element of $\mathbb{G}_{2p+1}^\times$ for all $\pm i$. To see this consider $\mathbb{Z}_9^\times$, which has order six rather than eight, since three has no multiplicative inverse in $\mathbb{Z}_9$.

$$D(r;t;s_1,s_2,s_3,s_4) = \begin{pmatrix} s_1 Re(q^{rt}) & s_2 Im(q^{rt}) \\ s_3 Im(q^{rt}) & s_4 Re(q^{rt}) \end{pmatrix} \tag{6.1}$$

$$E(r;t;s_1,s_2,s_3,s_4) = \begin{pmatrix} s_1 Im(q^{rt}) & s_2 Re(q^{rt}) \\ s_3 Re(q^{rt}) & s_4 Im(q^{rt}) \end{pmatrix} \tag{6.2}$$

where the s_i take values ± 1 and satisfy $s_1 s_2 s_3 s_4 = -1$ such that the associated F-matrices are orthogonal.

In addition, we define the matrices $G(r,\kappa)$, $H(r,\kappa)$ and $H'(r,\kappa)$ from the following function:

$$J(i,j;r,\kappa) = \frac{2^{\zeta(i,j)}\kappa}{\sqrt{2p+1}} \left(q^r\right)^{ij} \tag{6.3}$$

where $\zeta(i,j) = \frac{2-\delta_{i0}-\delta_{j0}}{2}$.

$G(r,\kappa)$ is a $p \times p$ matrix with entries

$$G(r,\kappa)_{i,j} = (-1)^{(i-1)(j-1)} Im\left(J(i,j;r,\kappa)\right) \tag{6.4}$$

whose indices run over $\{1,\ldots,p\}$. $H(r,\kappa)$ and $H'(r,\kappa)$ are $(p+1)\times(p+1)$ matrices with entries

$$H(r,\kappa)_{i,j} = (-1)^{ij} Re\left(J(i,j;r,\kappa)\right) \tag{6.5}$$

$$H'(r,\kappa)_{i,j} = (-1)^{\delta_{i0}+\delta_{j0}+1} H(r,\kappa)_{i,j}$$

whose indices run over $\{0,\ldots,p\}$. We note that the matrices $G(r)$ and $H^{(\prime)}(r)$ are orthogonal provided that r is an odd integer relatively prime to $2p+1$.

Arithmetic Data

Before we give the specific $6j$ symbols, we make a few general remarks. First, the order of the entries in the F-matrices respects the order we specified above: $(\mathbf{1}, \epsilon, \phi_i, \psi_\pm)$, where $i = 1,\ldots,p$. Second, we will actually not give the values $F_{abc}^{d;ef}$ individually, but instead give the F-matrices F_{abc}^{d}.

In the basis we use to give the F-matrices, we have the following property

$$F^a_{bcd} = \left(F^d_{abc}\right)^T \ . \tag{6.6}$$

In addition, it is implicitly assumed that the label d of F^d_{abc} is in the tensor decomposition of $a \otimes b \otimes c$. This allows us to only specify a reduced set of F-matrices, while the others can be deduced by using this relation. We note that often, but not not always, the F-matrices are symmetric. In addition, all the F-symbols are real in our basis.

We first observe that from (Natale and Plavnik, 2014) that for fixed p the fusion rules for $\{1, \epsilon, \psi_i\}$ give the tensor structure for $Rep(D_{2p+1})$. $K_0(\mathcal{C})$ is then a $\mathbb{Z}_2$-extension of $K_0(Rep(D_{2p+1}))$. We will proceed by building subcategories Grothendieck equivalent to $\mathbb{Z}_2$ and $Rep(D_{2p+1})$ before finally specifying F-matrices which correspond to the $\sqrt{2p+1}$ objects. If indices are omitted all objects are assumed the same.

First, if one or more of the labels a, b, c, d equals $\mathbf{1}$, $F^d_{abc} = (1)$. Next we have $\{\mathbf{1}, \epsilon\}$ equivalent to $\mathbb{Z}_2$ and the action of ϵ on the objects ϕ_i is determined by

$$F^{\epsilon}_{\epsilon\epsilon\epsilon} = (1) \qquad F^{\phi_i}_{\epsilon\phi_i\epsilon} = (1) \qquad F^{\phi_i}_{\epsilon\epsilon\phi_i} = (-1) \tag{6.7}$$

$$F^{\phi_k}_{\epsilon\phi_i\phi_j} = (-1^{j \bmod 2}) \qquad (j \leq i) \tag{6.8}$$

$$F^{\phi_k}_{\epsilon\phi_i\phi_j} = (-1^{j \bmod 2}) \qquad (j > i \wedge \kappa = g(i+j)) \tag{6.9}$$

$$F^{\phi_k}_{\epsilon\phi_i\phi_j} = (-1^{(j-1) \bmod 2}) \qquad (j > i \wedge \kappa = g(i-j)) \tag{6.10}$$

The rest of the F-matrices for the category $Rep(D_{2p+1})$ are given by:

$$F^{\phi}_{\phi\phi\phi} = C \tag{6.11}$$

$$F^{\phi_j}_{\phi_i\phi_j\phi_i} = B \qquad (j \neq i) \tag{6.12}$$

$$F^{\phi_j}_{\phi_i\phi_i\phi_j} = A(1, 1, -1^{(j-i+1) \bmod 2}, -1^{(j-i) \bmod 2}) \qquad (j \neq i) \tag{6.13}$$

$$F^{\phi_l}_{\phi_i\phi_j\phi_k} = (1) \qquad (\text{O.W}) \tag{6.14}$$

Proposition 6.2.2. *There are no other solutions to the pentagon equations for $Rep(D_{2p+1})$ which extend to solutions for $\mathfrak{so}(2p+1)_2$.*

Proof. By (Naidu and Rowell, 2011) all fusion categories $\mathcal{C}$ for $Rep(D_{2p+1})$ are group theoretical and thus Morita equivalent to a pointed fusion category $\mathcal{D}$ of the form (D_{2p+1}, ω) where $\omega \in H^3(D_{2p+1}, \mathbb{C}^\times)$, the group of $2p+1$ roots of unity. By (Ostrik, 2003) we know that (left) module categories over D_{2p+1} are parameterized by (H, ζ) where $H \leq G$ such that $\omega|_H = 1$ and $\zeta \in H^2(H, \mathbb{C}^\times)$.

In our case we have that since the universal grading of $\mathfrak{so}(2p+1)_2$ is $\mathbb{Z}_2$, we have that $\{\psi_+, \psi_-\}$ are indecomposable $\mathbb{Z}_2$ module categories over $Rep(D_{2p+1})$. We can count which $Rep(D_{2p+1})$ categories have $\mathbb{Z}_2$ module categories by looking at which D_{2p+1} categories have $\mathbb{Z}_2$ module categories. Since $H^3(D_{2p+1}, \mathbb{C}^\times) \cong \mathbb{Z}_{2p+1}$ the only ω which fixes the $\mathbb{Z}_2$ subgroup of D_{2p+1} is $\omega = 1$ and its co-homology is $\mathbb{Z}_2$. Thus only $Rep(D_{2p+1})$ has indecomposable $\mathbb{Z}_2$ module categories and there are two of them. □

We now specify the F-matrices that are labeled by $\psi_\pm$ and start with the F-matrices whose labels consist of only $\psi_\pm$.

$$F^{\psi}_{\psi\psi\psi} = H(r, \kappa) \tag{6.15}$$

$$F^{\psi_\mp}_{\psi_\pm,\psi_\mp,\psi_\pm} = -H(r, \kappa) \tag{6.16}$$

$$F^{\psi_\mp}_{\psi_\pm\psi_\pm\psi_\mp} = H'(r, \kappa) \tag{6.17}$$

$$F^{\psi_\mp}_{\psi_\pm\psi_\pm\psi_\pm} = \pm G(r, \kappa) \tag{6.18}$$

There are three classes of F-matrices involving two labels that are $\psi_\pm$. The first class of this type is

$$F^{\psi_\pm}_{\phi_i\psi_\pm\phi_j} = \mp(-1)^{ij} D(r; ij; -1, -1^{(i+j)}, -1^{(i+j)}, 1) \tag{6.19}$$

$$F^{\psi_\mp}_{\phi_i\psi_\pm\phi_j} = -(-1)^{ij} E(r; ij; -1^{(i+j)}, 1, 1, -1^{(i+j+1)}) \tag{6.20}$$

where i, j may or may not be equal. The second class of F-matrices with two labels equal to $\psi_\pm$ and $i \neq j$ is

$$
\begin{aligned}
F^{\psi_\pm}_{\phi_i\phi_i\psi_\pm} &= A(1,1,1,-1) \\
F^{\psi_\pm}_{\phi_i\phi_j\psi_\pm} &= A(\pm 1,\pm 1,1,-1) && (i-j) \bmod 2 = 0 \\
F^{\psi_\pm}_{\phi_i\phi_j\psi_\pm} &= A(1,-1,\pm 1,\pm 1) && (i-j) \bmod 2 = 1 \\
F^{\psi_\mp}_{\phi_i\phi_i\psi_\pm} &= A(-1,-1,(-1)^i,(-1)^{(i+1)}) \\
F^{\psi_\mp}_{\phi_i\phi_j\psi_\pm} &= (-1)^i A((-1)^{(i-j+1)},-1,(-1)^{(i-j)},-1) && i < j \\
F^{\psi_\mp}_{\phi_i\phi_j\psi_\pm} &= (-1)^i A((-1)^{(i-j)},1,(-1)^{(i-j)},-1) && i > j
\end{aligned}
$$

Finally, the third class of F-matrices with two labels equal to ψ_+ or ψ_- is

$$
F^{\psi_\pm}_{\epsilon\phi_i\psi_\pm} = F^{\psi_\mp}_{\epsilon\phi_i\psi_\pm} = F^{\psi_\pm}_{\epsilon\psi_\pm\phi_i} = F^{\phi_i}_{\epsilon\psi_\pm\psi_\pm} = F^{\phi_i}_{\epsilon\psi_\pm\psi_\mp} = \qquad (1)
$$

$$
F^{\psi_\mp}_{\epsilon\psi_\pm\phi_i} = F^{\psi_\pm}_{\epsilon\epsilon\psi_\pm} = F^{\psi_\pm}_{\epsilon\psi_\pm\epsilon} = (-1)
$$

With these symbols, and the general rules described above, we have exhausted all the F-matrices.

6.2.3 R-Matrices

Given our solutions to the pentagon equations, one would anticipate it is not too complicated to construct solutions to the hexagon equations as well. There is however a twist to the problem which we will comment on after stating the R-symbols.

Given a solution to the hexagon equations one always has a second solution given by inverses. This simply corresponds to a choice of R and R^{-1} in (4.20) and (4.21). We may

also obtain another solution by replacing all R-symbols R^c_{ab}, such that a and b are both one of $\psi_\pm$ by $-R^c_{ab}$. However, this is monoidally equivalent to our original solution via the automorphism $\psi_\pm \to \psi_\mp$. This gives us that for any p, solutions to the pentagon and hexagon equations can be uniquely identified by the tuple (p, r, κ, λ), where $\lambda = \pm 1$ indicates whether or not one is referring to the R-symbols given below or their inverses. We then provide only one such solution.

We specify the R-symbols using a form inspired by conformal field theory. Namely, for each simple object type, we introduce the scaling dimensions h_a which determine the R-symbols up to a sign

$$R^c_{ab} = (\sigma_1)^c_{ab}(\sigma_2)^c_{ab}(-1)^{h_a+h_b-h_c},$$

where $(\sigma_1)^c_{ab} = \pm 1$ and $(\sigma_2)^c_{ab} = \pm 1$. Below we specify the scaling dimensions h_a for all the simple objects, as well as the signs $(\sigma_1)^c_{ab}$ and $(\sigma_2)^c_{ab}$.

The scaling dimensions are given by

$$\begin{aligned}
h_{\mathbf{1}} &= 0 \\
h_\epsilon &= 1 \\
h_{\phi_i} &= \frac{ri(2p+1-i)}{2(2p+1)} \\
h_{\psi_+} &= \frac{r(p+\kappa\, s_p + (p+(r+1)/2|r)\, s_r + 2)}{8} \\
h_{\psi_-} &= \frac{r(p+\kappa\, s_p + (p+(r+1)/2|r)\, s_r + 6)}{8}
\end{aligned}$$

where

$$s_p = \begin{cases} +1 & (p \bmod 4) = 1 \vee (p \bmod 4) = 2 \\ -1 & (p \bmod 4) = 0 \vee (p \bmod 4) = 3 \end{cases}$$

$$s_r = \begin{cases} +1 & (r \bmod 8) = 3 \vee (r \bmod 8) = 5 \\ -1 & (r \bmod 8) = 1 \vee (r \bmod 8) = 7 \end{cases}$$

The notation $(j|n)$ denotes the Jacobi symbol, which is defined for j an integer, and n a positive, odd integer.

To completely specify the R-symbols we still need to specify the signs $(\sigma_1)^c_{ab}$ and $(\sigma_2)^c_{ab}$.

We start with $(\sigma_1)^c_{ab}$, and note that for the solution presented we have the property $(\sigma_1)^a_{bc} = (\sigma_1)^c_{ab}$, which implies that we only need to specify a limited set of these symbols. The non-trivial symbols come in two classes. The first class is

$$(\sigma_1)^{\psi_\pm}_{\phi_i\psi_\pm} = (\sigma_1)^{\psi_\mp}_{\phi_i\psi_\pm} = \begin{cases} -1 & ((i \bmod 4) = (0 \vee 1)) \wedge (p \bmod 2) = 1 \\ -1 & ((i \bmod 4) = (1 \vee 2)) \wedge (p \bmod 2) = 0 \\ +1 & \text{otherwise.} \end{cases}$$

The second class of symbols is

$$(\sigma_1)^{\phi_{g(i+j)}}_{\phi_i\phi_j} = \begin{cases} -1 & (-1)^i = (-1)^j = -1 \\ +1 & \text{otherwise.} \end{cases} . \tag{6.21}$$

The symbols $(\sigma_1)^c_{ab}$ that are not specified by the rules above, are all equal to $+1$.

Finally, the signs $(\sigma_2)^c_{ab}$ also satisfy the property $(\sigma_2)^a_{bc} = (\sigma_2)^c_{ab}$ and there is only one class of non-trivial signs of type, namely

$$(\sigma_2)^{\psi_\mp}_{\epsilon\psi_\pm} = (\sigma_2)^{\psi_\mp}_{\phi_i\psi_\pm} = (-1)^{\frac{r-1}{2}} .$$

The symbols $(\sigma_2)^c_{ab}$ that not specified by the rules above, are all equal to $+1$.

With this, we have completely specified one of the solutions of the hexagon equations, from which the others can be constructed easily.

Remark 6.2.3. We close by commenting briefly on the appearance of the Jacobi symbols in the expression for the R-symbols. One can think of the list of hexagon equations as labeled by the different F-symbols, namely the one appearing on the left hand side of the symbolic form $RFR = \sum FRF$. To derive the form of $h_{\psi_\pm}$ above, consider the hexagon equation

associated to $F^{\psi_+;\mathbf{11}}_{\psi_+\psi_+\psi_+}$. The sum on the right hand side gives rise to a quadratic Gauss sum (because one needs to sum over all ϕ_i (as well as the identity)), which are closely related to the Jacobi symbols. This leads to their presence in the scaling dimensions $h_{\psi_\pm}$.

The Jacobi symbols $(j|n)$ are defined for positive odd integers n and arbitrary integers j. Interestingly, we observed that the these Jacobi symbols can be written in terms of the matrices $H(r,\kappa)$ and $G(r,\kappa)$ (whose sizes depend on p). In particular, one can show

$$(r|2p+1) = \det(H(2r+2p+1))\det(G(2r+2p+1)).$$

This gives an analytic function of r, that goes through all the Jacobi symbols, defined for r integer. We note that Eisenstein also has such a function, $(q|p) = \prod_{n=1}^{(p-1)/2} \frac{\sin(2\pi qn/p)}{\sin(2\pi n/p)}$. The difference between the function that we found, is that the values ± 1 are the extrema of the function while this is certainly not the case for Eisenstein's function.

6.2.4 Modular data for $\mathfrak{so}(2p+1)_2$ Modular Systems

All of our F-Matrices are manifestly unitary, thus all of our categories admit at least one spherical structure. Since they're also braided, they must be ribbon and so we can use the formula (4.24). Our formula (4.23) is defined for any braided category, however one can obtain the more common form by taking DSD. We present the latter to more easily demonstrate that we obtain the same modular result as (Gelaki et al., 2009). However, we also provide an explicit classification at the fusion level.

Pivotal structures on modular categories are in bijective correspondence with the group of invertible objects and spherical structures with the maximal Abelian 2-subgroup(Bruillard et al., 2016). Since the group of invertible objects is $\mathbb{Z}_2$, there are exactly two pivotal structures for each category, both of which are spherical. It is straightforward to compute that for $\mathbf{1}$, ϵ, and ϕ_i, all pivotal coefficients must be 1. We also have $\epsilon_{\psi_+} = \epsilon_{\psi_-} = \pm 1$, the choice of which switches the sign of $q_{\psi_\pm}$.

Below, we present the modular data for solutions with positive quantum dimensions. It's worth noting that for all p and r the pivotal coefficients which yield positive quantum dimensions for $\kappa = 1$ are not those that yield positive quantum dimensions for $\kappa = -1$. This can be seen from noting the effect of κ on (4.19).

We then have that, for modular categories corresponding to the solution indexed by (p, r, κ, λ),

$$S_{\mathbf{11}} = 1 \qquad S_{\epsilon\epsilon} = 1 \qquad S_{\phi_i\phi_j} = 4\cos\left(\frac{2ijr}{2p+1}\right) \tag{6.22}$$

$$S_{\mathbf{1}\epsilon} = 1 \qquad S_{\epsilon\phi_i} = 2 \qquad S_{\phi_i\psi_\pm} = 0 \tag{6.23}$$

$$S_{\mathbf{1}\phi_i} = 2 \qquad S_{\epsilon\psi_\pm} = -\kappa(2|2p+1)q_{\psi_\pm} \qquad S_{\psi_\pm\psi_\pm} = -\kappa(2|2p+1)q_{\psi_\pm} \tag{6.24}$$

$$S_{\mathbf{1}\psi_\pm} = q_{\psi_\pm} \qquad S_{\psi_\pm\psi_\mp} = \kappa(2|2p+1)q_{\psi_\mp} \tag{6.25}$$

For the twists we have

$$\theta_\epsilon = 1 \qquad \theta_{\phi_i} = \exp(\frac{i^2 r}{2p+1})^\lambda \qquad \theta_{\psi_\pm} = (-1)^{\frac{\kappa\lambda+1}{2}}\Delta_\pm \tag{6.26}$$

where

$$\Delta_\pm = \begin{cases} \pm\sqrt{\kappa\lambda(r|2p+1)} & (2p+1) \mod 4 = 1 \\ \pm\sqrt{\kappa\lambda\imath(r|2p+1)} & (2p+1) \mod 4 = 3 \end{cases}$$

6.3 Monoidal Equivalence of $\mathfrak{so}(2p+1)_2$ Fusion Systems

In this section we introduce those gauge invariants which we will use to classify monoidally inequivalent solutions for a given p using the method of Section 5.2.2. At the level of gauge equivalence it is easy to see that two solutions (r, κ) and (r', κ') are gauge equivalent if and only if $r = r'$ and $\kappa = \kappa'$. Our goal then is to find a set $\{l_1, \ldots, l_n\}$ of $Aut(N)$-invariant linear combination of G-invariant monomials as defined in Theorem 5.1.2 which determine the monoidal classes. We then determine the number of monoidal equivalence classes for a given p. For all pairs (r, κ) as defined in Section 6.2.2 let $F_{(r,k)}$ be the evaluation map as defined for Theorem 5.1.2.

6.3.1 Determining Equivalence

We first note that the adjoint category is basically useless to us.

Proposition 6.3.1. *The monoidal classes of solutions specified in 6.2.2 cannot be distinguished by $Aut(N) \ltimes G$-invariants coming from $Rep(D_{2p+1})$ subcategories.*

Proof. For fixed p, the only F-matrix entries which differ are those in which r arises, i.e., those specified by the D, E, $H^{(\prime)}$, and G matrices. These come from F-matrices involving at least one $\psi_\pm$ and so do not come from the $Rep(D_{2p+1})$ subcategory. So if l is an $Aut(N) \ltimes G$-invariant which comes from $Rep(D_{2p+1})$, i.e., only involves the objects 1, ϵ, and $\phi_i, i = 1, \dots p$ then

$$F_{(r,k)}(l) = F_{(r',k')}(l).$$

□

From here, we can narrow our search even further. From (6.15)-(6.20), the entries of

$$F^{\psi_\pm}_{\phi_i\psi_\pm\phi_j}, F^{\psi_\mp}_{\phi_i\psi_\pm\phi_j}, F^{\psi_\mp}_{\psi_\pm\psi_\mp\psi_\mp}, \text{ and } F^{\psi_\mp}_{\psi_\pm\psi_\mp\psi_\pm}$$

can all be written in some fixed way as scalar multiples of entries in $F^{\psi_\pm}_{\psi_\pm\psi_\pm\psi_\pm}$ or $F^{\psi_\mp}_{\psi_\pm\psi_\pm\psi_\pm}$. These relationships are preserved under permutation. The entries of $F^{\psi_\pm}_{\psi_\pm\psi_\pm\psi_\pm}$ and $F^{\psi_\mp}_{\psi_\pm\psi_\pm\psi_\pm}$ are given by the matrices $H(r,\kappa)$ and $\pm G(r,\kappa)$ respectively.

Proposition 6.3.2. *Let $F_{(r,\kappa)}, F'_{(r',\kappa')}$ be solutions to (4.15)-(4.17) parameterized by (r,κ) and (r',κ'). Then $(r,\kappa) = (r',\kappa')$ if and only if $H(r,\kappa) = H(r',\kappa')$.*

Proof. The direction $(r,\kappa) = (r',\kappa') \Rightarrow H(r,\kappa) = H(r',\kappa')$ is obvious, so assume $H(r,k) = H(r',k')$. $H(r,\kappa)$ determines $G(r,\kappa)$ up to a sign and

$$(F_{(r,k)})^{\psi_\pm}_{\psi_\pm\psi_\pm\psi_\pm} = (F_{(r',k')})^{\psi_\pm}_{\psi_\pm\psi_\pm\psi_\pm} \qquad \text{and}$$
$$(F_{(r,k)})^{\psi_\mp}_{\psi_\pm\psi_\pm\psi_\pm} = \pm(F_{(r',k')})^{\psi_\mp}_{\psi_\pm\psi_\pm\psi_\pm}.$$

This corresponds to a choice $\psi_\pm$, which is irrelevant to (r,κ), so this implies $(r,\kappa) = (r',\kappa')$.

□

Corollary 6.3.3. $(r,\kappa) = (r',\kappa')$ *if and only if*

$$Diag(H(r,\kappa)) = Diag(H(r',\kappa')).$$

Proof. $H(r,\kappa)_{0,0}$ determines κ and $H(r,\kappa)_{1,1} \propto Re(q^r)$ narrows r to one of two choices. Taken modulo $2p+1$ only one of r and $-r$ is odd, and so $H(r,\kappa)_{1,1}$ determines $H(r,\kappa)$. $\square$

By inspection, we see that

$$(F_{(r,k)})^{\psi_\pm;\phi_i\phi_i}_{\psi_\pm\psi_\pm\psi_\pm}(F_{(r,k)})^{\phi_i;\psi_\pm\psi_\pm}_{\psi_\pm\phi_i\psi_\pm} = H(r,k)_{i,i}$$

so define

$$X_p(r,\kappa,i) = \frac{\sqrt{2p+1}}{2^{\zeta(i,i)}} \sum_{j\in\{+,-\}} \left((F_{(r,k)})^{\psi_j;\phi_i\phi_i}_{\psi_j\psi_j\psi_j}(F_{(r,k)})^{\phi_i;\psi_j\psi_j}_{\psi_j\phi_i\psi_j} \right)$$

and the $(p+1)$-tuple

$$X_p(r,\kappa) = (X_p(r,\kappa,i)|i = 0,\ldots,p).$$

Corollary 6.3.4. $X_p(r,\kappa)$ *uniquely determines the values for all G-invariant monomials and thus all $Aut(N) \ltimes G$-invariant monomials.*

Corollary 6.3.5. *The permutation $\psi_+ \to \psi_-$ acts as a gauge transformation.*

There is a natural action of $z \in \mathbb{G}^\times_{2p+1}$ on $X(r,\kappa)$ given by

$$\begin{aligned} z \cdot X_p(r,\kappa) &= (z \cdot X_p(r,\kappa,i)|i = 0,\ldots p) \\ &= (X_p(r,\kappa,g(z*i))|i = 0,\ldots,p) \\ &= (X_p(rz^2,\kappa,i)|i = 0,\ldots p) \\ &= X_p(rz^2,\kappa). \end{aligned}$$

Theorem 6.3.6. *Fixed p and let (r,κ) and (r',κ) be solutions as defined in 6.2.2. The following are equivalent:*

1. *(r,κ) and (r',κ) are monoidally equivalent,*

2. *there exists $z \in \mathbb{G}_{2p+1}^{\times}$ such that $X(r',\kappa) = X(rz^2,\kappa)$, and*

3. *there exists $z \in \mathbb{G}_{2p+1}^{\times}$ such that $g(r') = g(rz^2)$.*

Proof. First assume that (r,κ) and (r',κ) are monoidally equivalent via some $z \in \mathbb{G}_{2p+1}^{\times}$. z acts on $F_{\psi_\pm\psi_\pm\psi_\pm}^{\psi_\pm}$ by

$$z \cdot F_{\psi_\pm\psi_\pm\psi_\pm}^{\psi_\pm;\phi_i\phi_j} = F_{\psi_\pm\psi_\pm\psi_\pm}^{\psi_\pm;\phi_{g(zi)}\phi_{g(zj)}} = H(rz^2,\kappa)$$

so

$$X_p(r',\kappa) = X_p(rz^2,\kappa).$$

This implies r' and rz^2 are congruent modulo $2p+1$ and so $g(r') = g(rz^2)$.

Assume $\exists z \in \mathbb{G}_{2p+1}^{\times}$ such that $g(r') = g(rz^2)$. Then, modulo $2p+1$, $r' \cong rz^2$ or $r' \cong -rz^2$, but regardless $X_p(r',\kappa) = X_p(rz^2,\kappa)$. By Corollary 6.3.4, for any $Aut \ltimes G$-invariant l,

$$F_{(r',\kappa)}(l) = F_{(r,\kappa)}(l).$$

Thus (r',κ) and (r,κ) are monoidally equivalent. □

6.3.2 Calculating The Number of Monoidal Equivalence Classes

It is not an easy task to calculate how many monoidally inequivalent sets of F-moves there are. Nevertheless we can construct a framework in which the question is seemingly simple. From the previous section we showed that the F-moves labelled by (r,κ) and (r',κ') are monoidally equivalent if and only if $\kappa = \kappa'$ and there exists a $z \in \mathbb{G}_{2p+1}^{\times}$ such that $r' = rz^2$. Thus we define

$$O_r \;=\; \{rz^2 | z \in \mathbb{G}_{2p+1}^{\times}\} \subseteq \mathbb{G}_{2p+1}^{\times}.$$

Each set contains elements such that the F-moves labelled by (r, κ) and (r', κ) are monoidally equivalent. Furthermore, (r, κ) and (r', κ) are monoidally equivalent if and only if $O_r = O_{r'}$. We realise that O_1 is a normal subgroup and that $O_r = rO_1$ are conjugacy classes. Thus the number of monoidally inequivalent F-moves must be twice (due to κ) $|\mathbb{G}^{\times}_{2p+1}|/|O_1|$.

It is possible to simplify this further by considering the map $f : \mathbb{G}^{\times}_{2p+1} \to O_1$ defined by $f(z) = z^2$. This map is clearly onto, i.e., the image of f is O_1. As we are dealing with the finite groups we can use the first isomorphism theorem and deduce that $|\mathbb{G}^{\times}_{2p+1}|/|O_1| = |ker(f)|$ where $ker(f)$ is the kernel of f and is the subgroup

$$ker(f) \quad = \quad \{r \in \mathbb{G}^{\times}_{2p+1} | r^2 = 1\}.$$

Thus the number of monoidally inequivalent F-moves is twice the number of elements of $\mathbb{G}^{\times}_{2p+1}$ that square to the identity (because of κ). We know that $|ker(g)| = 2^m$ for some $m \in \mathbb{N}$.

If $r \in \mathbb{Z}^{\times}_{2p+1}$ satisfies $r^2 = 1$ or -1 then $r \in \mathbb{G}^{\times}_{2p+1}$ satisfies $r^2 = 1$. We can then deduces that

$$|ker(f)| \quad = \quad \frac{1}{2}\left(|\{r \in \mathbb{Z}^{\times}_{2p+1} | r^2 = 1\}| + |\{r \in \mathbb{Z}^{\times}_{2p+1} | r^2 = -1\}|\right).$$

Firstly, consider the prime decomposition $2p + 1 = p_1^{a_1} ... p_l^{a_l}$ where the p_i are prime and $a_i > 0$. We define sets

$$\begin{aligned} B_1 \quad &= \quad \{r \in \mathbb{Z}^{\times}_{2p+1} | r^2 = 1\}, \\ B_2 \quad &= \quad \{r \in \mathbb{Z}^{\times}_{2p+1} | r^2 = -1\}. \end{aligned}$$

We note that as B_1 is also a group and that its order is precisely 2^l. Next, suppose that $r, r' \in B_2$ it follows that $rr' \in B_1$, simultaneously if $r \in B_2$ and $r' \in B_1$ then $rr' \in B_2$. Thus we have that the cardinality of B_2 is zero or $|B_2| = |B_1| = 2^l$. For $|B_2|$ to be non-zero we

must have that there exists b_i such that $b_i^2 \equiv -1 \bmod p_i^{a_i}$ for all i.

From this we can deduce the number of monoidal classes

$$\#\text{monoidal classes} \;=\; \begin{cases} 2^{l+1} & \text{if } \exists b \text{ s.t. } b^2 \equiv -1 \bmod (2p+1), \\ 2^l & \text{otherwise,} \end{cases} \tag{6.27}$$

where l is the number of primes that divide $(2p+1)$. Here we have used that $\mathbb{Z}^\times_{2(2p+1)} \cong \mathbb{Z}^\times_{2p+1}$.

Remark 6.3.7. As a sanity check on this, following (Ardonne et al., 2016, Theorem 1.1), we note that for fixed p the number of metaplectic fusion categories is always less than or equal to the number of metaplectic modular categories.

6.3.3 Constructing $Aut \ltimes G$-Invariants

Ideally, we would like to find a single $Aut \ltimes G$-invariant function capable of distinguishing monoidal classes. Given the form of $X_p(r,\kappa)$, a naive guess is something of the form

$$\begin{aligned} Y_p(r,\kappa,n) &= \sum_{\substack{i\in\mathbb{G}_{2p+1},\\ j\in\{+,-\}}} \frac{\sqrt{2p+1}}{2^{\zeta(i,i)}} \left(F^{\psi_j;\phi_i\phi_i}_{\psi_j\psi_j\psi_j} F^{\phi_i;\psi_j\psi_j}_{\psi_j\phi_i\psi_j}\right)^n \\ &= \sum_{i=0}^{p} \left[(-1)^{i^2}\kappa\cos\left(\frac{\pi r i^2}{2p+1}\right)\right]^n \\ &= \kappa\sum_{i=0}^{p}(-1)^{i^2}\cos^n\left(\frac{\pi r i^2}{2p+1}\right) \end{aligned} \tag{6.28}$$

for some odd $n \in \mathbb{N}$. This leads us to our first conjecture.

Conjecture 6.3.8. *Fix p and let (r,κ) and (r',κ') parameterize solutions to (4.15)-(4.17) for monoidally inequivalent categories. Then there exists $n \in \mathbb{N}$ such*

$$Y_p(r,\kappa,n) \neq Y_p(r',\kappa',n).$$

The game is then, for fixed p, to find an n which distinguishes between monoidal classes. $n = 1$ seems to work in most cases, but the first example for which it fails is $p = 12$ which requires $n = 5$.

To come up with a way of picking a sufficiently large n for arbitrary p, we first simplify (6.28) by expanding the $\cos^n(\ldots)$ term using the identity

$$\cos^n(\theta) = \frac{2}{2^n} \sum_{l=0}^{\frac{n-1}{2}} \binom{n}{l} \cos((n-2l)\theta)$$

when n is odd. Letting $\theta = \frac{\pi r i^2}{2p+1}$, we can use the sum rule for cosines ($\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$) to simplify:

$$\begin{aligned}\cos\left((n-2l)\left(\frac{\pi r i^2}{2p+1}\right)\right) &= \cos\left(\frac{n\pi r i^2}{2p+1} - \frac{2\pi r l i^2}{2p+1}\right) \\ &= \cos\left(\frac{n\pi r i^2}{2p+1}\right)\cos\left(\frac{2\pi r l i^2}{2p+1}\right) + \sin\left(\frac{n\pi r i^2}{2p+1}\right)\sin\left(\frac{2\pi r l i^2}{2p+1}\right)\end{aligned}$$

If $n = 2p+1$, then

$$\cos\left((2p+1-2l)\left(\frac{\pi r i^2}{2p+1}\right)\right) = (-1)^{i^2}\cos\left(\frac{2\pi r l i^2}{2p+1}\right)$$

and we have that

$$\begin{aligned}(-1)^{i^2}\cos^{2p+1}\left(\frac{\pi r i^2}{2p+1}\right) &= \frac{2(-1)^{i^2}}{2^{2p+1}} \sum_{l=0}^{\frac{2p+1-1}{2}} \binom{2p+1}{l} \cos\left((2p+1-2l)\left(\frac{\pi r i^2}{2p+1}\right)\right) \\ &= \left(\frac{1}{4^p}\right) \sum_{l=0}^{p} \binom{2p+1}{l} \cos\left(\frac{2\pi r l i^2}{2p+1}\right).\end{aligned}$$

We can substitute back in to (6.28) to obtain

$$\begin{aligned}\kappa \sum_{i=0}^{p} (-1)^{i^2}\cos^{2p+1}\left(\frac{\pi r i^2}{2p+1}\right) &= \kappa\left(\frac{1}{4^p}\right)\sum_{i=0}^{p}\left[\sum_{l=0}^{p}\binom{2p+1}{l}\cos\left(\frac{2\pi r l i^2}{2p+1}\right)\right] \\ &= \kappa\left(\frac{1}{4^p}\right)\sum_{l=0}^{p}\left[\binom{2p+1}{l}\left(\sum_{i=0}^{p}\cos\left(\frac{2\pi r l i^2}{2p+1}\right)\right)\right] \qquad (6.29)\end{aligned}$$

which can be considered as the real part of

$$Z_p(r,\kappa) = \kappa\left(\frac{1}{4^p}\right)\sum_{l=0}^{p}\left[\binom{2p+1}{l}\left(\sum_{i=0}^{p}\left(q^{ri^2}\right)^l\right)\right]. \qquad (6.30)$$

By definition q^{ri^2} is a non-degenerate quadratic form on $\mathbb{Z}_{2p+1}$ and the terms in parentheses are incomplete Gauss sums.

That $n = 2p + 1$ is sufficient can be checked by computer fairly quickly for p between 1 and 1000. This then leads to our second conjecture, which refines the first:

Conjecture 6.3.9. *Fix p and let (r, κ) and (r', κ') parameterize solutions to (4.15)-(4.17) for monoidally inequivalent categories. Then*

$$Y_p(r, \kappa, 2p + 1) \neq Y_p(r', \kappa', 2p + 1).$$

6.4 Examples

In this section we explicitly compute the classifying invariants for a number of "small" categories. The case of $\mathfrak{so}(3)_2$ is known to be Grothendieck equivalent to $\mathfrak{su}(2)_4$ and both the fusion and modular structures are fully classified (see (Kazhdan and Wenzl, 1993)). Beyond this, there are several guideposts:

- The classification of weakly integral modular categories of dimension $4m$ is given in (Bruillard et al., 2014). This contains those $\mathfrak{so}(2p+1)_2$ of our family for which $2p+1$ is square free.

- The classification of Integral modular categories of dimension $4q^2$ is given in (Bruillard et al., 2013) where q is prime.

- Explicit formulae for the modular data of $\mathbb{Z}_2$-equivariantizations of Tambara-Yamagami categories is given in (Gelaki et al., 2009) to which our categories are Grothendieck equivalent.

6.4.1 $\mathfrak{so}(3)_2$

For $\mathfrak{so}(3)_2$, $\mathbb{G}_3^\times = \{1\}$. Thus we have only one two dimensional simple object for which the automorphism group is trivial. We have two solutions and two fusion categories. Up to

permutation there are four modular categories, one for each of the four possible T-matrices coming from (6.26):

$$diag(1,1,-(-1)^{1/4},(-1)^{1/4},-(-1)^{1/3})$$
$$diag(1,1,(-1)^{3/4},-(-1)^{3/4},(-1)^{2/3})$$
$$diag(1,1,(-1)^{3/4},-(-1)^{3/4},-(-1)^{1/3})$$
$$diag(1,1,-(-1)^{3/4},(-1)^{1/4},(-1)^{2/3})$$

Corresponding to solutions $(1,1,1,1)$, $(1,1,1,-1)$, $(1,1,-1,1)$ and $(1,1,-1,-1)$ respectively. This is in agreement with (Gelaki et al., 2009). In this case the two braidings for each fusion category are inequivalent.

6.4.2 $\mathfrak{so}(5)_2$

There are two two dimensional objects and $\mathbb{G}_5^\times \cong \mathbb{Z}_2$. Both elements square to the identity and so there are four fusion categories from (6.27). This can be seen from the sets

$$X_2(1,1) = \left\{1, \frac{1}{4}\left(-1-\sqrt{5}\right), \frac{1}{4}\left(-1-\sqrt{5}\right)\right\}, \text{ and}$$
$$X_2(3,1) = \left\{1, \frac{-1}{4}\left(1-\sqrt{5}\right), \frac{-1}{4}\left(1-\sqrt{5}\right)\right\},$$

From (Gelaki et al., 2009) there are also four modular categories, corresponding to solutions $(1,1,1,1)$, $(1,1,-1,1)$, $(3,1,1,1)$, and $(3,1,-1,1)$ with T-matrices

$$diag(1,1,-1,1,-(-1)^{1/5},(-1)^{4/5}),$$
$$diag(1,1,\imath,-\imath,-(-1)^{1/5},(-1)^{4/5}),$$
$$diag(1,1,-\imath,\imath,-(-1)^{3/5},(-1)^{2/5}), \text{ and}$$

$$diag(1, 1, -1, 1, -(-1)^{3/5}, (-1)^{2/5})$$

In this case the braidings corresponding to $\lambda = \pm 1$ are equivalent, which is easy to see from (6.26).

6.4.3 $\mathfrak{so}(7)_2$

There are three 2D objects and $G_7^\times \cong \mathbb{Z}_3$. We have that

$$\begin{aligned}
X_3(1,1) &= \left\{1, -\cos\left(\frac{\pi}{7}\right), -\sin\left(\frac{\pi}{14}\right), \sin\left(\frac{3\pi}{14}\right)\right\}, \\
X_3(3,1) &= \left\{1, -\sin\left(\frac{\pi}{14}\right), \sin\left(\frac{3\pi}{14}\right), -\cos\left(\frac{\pi}{7}\right)\right\}, \text{ and} \\
X_3(5,1) &= \left\{1, \sin\left(\frac{3\pi}{14}\right), -\cos\left(\frac{\pi}{7}\right), -\sin\left(\frac{\pi}{14}\right)\right\}.
\end{aligned}$$

These are all clearly related via permutation. Computing the modular data for our solutions, we find that there are four modular categories distinguished by their T-matrices

$$\begin{aligned}
&diag(1, 1, -(-1)^{1/4}, (-1)^{1/4}, -(-1)^{1/7}, (-1)^{4/7}, (-1)^{2/7}) \\
&diag(1, 1, (-1)^{3/4}, -(-1)^{3/4}, (-1)^{6/7}, -(-1)^{3/7}, -(-1)^{5/7}) \\
&diag(1, 1, (-1)^{3/4}, -(-1)^{3/4}, -(-1)^{1/7}, (-1)^{4/7}, (-1)^{2/7}), \text{ and} \\
&diag(1, 1, (-1)^{1/4}, -(-1)^{1/4}, (-1)^{6/7}, -(-1)^{3/7}, -(-1)^{5/7})
\end{aligned}$$

corresponding to solutions $(3, 1, 1, 1)$, $(3, 1, 1, -1)$, $(3, 1, -1, 1)$, and $(3, 1, -1, -1)$ respectively.

6.4.4 $\mathfrak{so}(9)_2$

For $\mathfrak{so}(9)_2$ there are four two dimensional objects, but we have that $\mathbb{G}_9 \cong \mathbb{Z}_3$ since there are three odd integers less than and co-prime to 9: 1, 5, and 7. These correspond to the S_4 subgroup generated by the cycle $\langle 124 \rangle$.

The sets $X_4(r,1)$ are then:

$$X_4(1,1) = \left\{ -\cos\left(\frac{\pi}{9}\right), \cos\left(\frac{4\pi}{9}\right), 1, \cos\left(\frac{2\pi}{9}\right) \right\},$$
$$X_4(5,1) = \left\{ \cos\left(\frac{4\pi}{9}\right), \cos\left(\frac{2\pi}{9}\right), 1, -\cos\left(\frac{\pi}{9}\right) \right\}, \text{ and}$$
$$X_4(7,1) = \left\{ \cos\left(\frac{2\pi}{9}\right), -\cos\left(\frac{\pi}{9}\right), 1, -\cos\left(\frac{4\pi}{9}\right) \right\}.$$

From this it's easy to see that $\langle 124 \rangle$ sends $X_4(5,1)$ to $X_4(1,1)$ and $\langle 142 \rangle$ sends $X_4(7,1)$ to $X_4(1,1)$. Thus there is only one equivalence class. Finally, as before there are four modular structures distinguished by their T-matrices:

$$diag(1, 1, -1, 1, -(-1)^{1/9}, (-1)^{4/9}, 1, -(-1)^{7/9}),$$
$$diag(1, 1, -1, 1, (-1)^{8/9}, -(-1)^{5/9}, 1, (-1)^{2/9}),$$
$$diag(1, 1, -\imath, \imath, -(-1)^{1/9}, (-1)^{4/9}, 1, -(-1)^{7/9}), \text{ and}$$
$$diag(1, 1, \imath, -\imath, (-1)^{8/9}, -(-1)^{5/9}, 1, (-1)^{2/9}).$$

CHAPTER 7

CONCLUSION

In this chapter, we summarize our results and outline directions for future work.

7.1 Summary of Results

In this thesis, we have developed a powerful tool for classifying fusion categories and, in turn, topological phases and anyonic systems. In particular, we have shown that when phrased correctly, the classification problem can be reduced to a version of Hilbert's fourteenth problem. As a showcase of the power of this technique, we used these invariants to refine the classification of $\mathfrak{so}(2p+1)_2$ categories, also called metaplectic categories, down to the fusion level.

The success of this approach is owed in no small part to formulating the problem of solving pentagon equations using the language of schemes and invariant theory. While it is obvious that this can be done, doing so is laborious and so this particular direction floundered. However, extending the work of (Davidovich et al., 2013) allowed us to construct a geometrical interpretation of the arithmetic data.

Once done, the results of (Mumford et al., 1994) serve to simplify the problem of determining when two fusion or modular categories, and thus two string-nets or anyonic systems, are different. Because of the structure of the gauge group and its action on the on the scheme $X(R, B)$, the algebra of invariant functions is finitely generated, and so there is a scheme whose points "are" equivalence classes of fusion categories. When the associated based ring (R, B) is multiplicity free, there is, modulo knowing which variables are zero, a fast algorithm for computing a generating set of invariants.

7.2 Future Work

First and foremost is the development of an algorithm for computing invariants in systems with multiplicity. While the algorithm for multiplicity-free invariants was discovered combinatorially, the existence of invariants for the multiplicity case were not apparent. Now that we know they do exist, the obvious question is: How do we find them?

From a physical stand point, the next frontier is nailing down the arithmetic formalism for describing domain walls and gapped boundaries in topological systems. The work of (Kitaev and Kong, 2011) and (Kong, 2014) outline the mathematical objects necessary to do this – specifically, algebra objects in fusion categories and module categories over fusion categories. However, it assumes one has possession of the necessary building blocks ahead of time. They are also both light on the details for the input data necessary to construct such models.

Related to this is a development of a physical interpretation for fusion category invariants. Our work implies the potential for observable invariant quantities at the level of quasi-particle fusion. This may be useful in studying models of anyons on boundaries where braiding is not allowed but fusion is. These arise in the study of quasi-particle excitations on gapped-boundaries.

The classification of metaplectic categories has been of interest for several years. One aspect of this was understanding the computational power of the $q_{\phi_i} = 2$ anyons and the computational complexity of their braid group representations. There is also interest in extending the results of (Finch et al., 2014) for one dimensional chains of $\mathfrak{so}(5)_2$ anyons to all $\mathfrak{so}(2p+1)_2$ theories. Here, the arithmetic data plays an important roll in the definition of the Hamiltonian. This is, in fact, how the author became involved in the $\mathfrak{so}(2p+1)_2$ problem and is an ongoing area for research.

Another, more purely mathematical, direction for future work is in using gauge invariants to prove the (S,T)-conjecture. That the S and T matrix entries are gauge invariant using

our formalism isn't a surprise since they are gauge invariant in the physical formalism. In that case, our work simply provides some extra meaning. However, our work shows that only a finite number of such invariant quantities are necessary.

APPENDIX

ARITHMETIC DATA FOR SOLUTIONS TO THE GROTHENDIECK RING FOR $Rep(D(S_3))$

A.1 Solution 1 – $Rep(D(S_3))$

A.1.1 F-Matrices

- $\mathrm{F}^{\alpha_+}_{\alpha_+,\alpha_-,\alpha_-} = \mathrm{F}^{\alpha_-}_{\alpha_-,\alpha_+,\alpha_+} = \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$

- $\mathrm{F}^{\alpha_-}_{\alpha_+,\alpha_-,\alpha_+} = \mathrm{F}^{\alpha_+}_{\alpha_-,\alpha_+,\alpha_-} = \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{\sqrt{2}}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{\sqrt{2}}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

- $\mathrm{F}^{\alpha_-}_{\alpha_+,\alpha_+,\alpha_-} = \mathrm{F}^{\alpha_+}_{\alpha_-,\alpha_-,\alpha_+} = \begin{pmatrix} \frac{1}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{\sqrt{2}}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$

- $\mathrm{F}^{\alpha_+}_{\alpha_+,\alpha_+,\alpha_+} = \mathrm{F}^{\alpha_-}_{\alpha_-,\alpha_-,\alpha_-} = \begin{pmatrix} \frac{1}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{\sqrt{2}}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

- $\mathrm{F}^{\alpha_+}_{\alpha_+,\alpha_+,\alpha_-} = \mathrm{F}^{\alpha_-}_{\alpha_+,\alpha_-,\alpha_-} = \mathrm{F}^{\alpha_+}_{\alpha_-,\alpha_+,\alpha_+} = \mathrm{F}^{\alpha_-}_{\alpha_-,\alpha_-,\alpha_+} = \begin{pmatrix} -\frac{i}{\sqrt{3}} & 0 & -\frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \\ 0 & \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} \\ -\frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} & 0 & \frac{i}{\sqrt{3}} \\ -\frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} & 0 \end{pmatrix}$

- $\mathrm{F}^{\alpha_-}_{\alpha_+,\alpha_+,\alpha_+} = \mathrm{F}^{\alpha_+}_{\alpha_+,\alpha_-,\alpha_+} = \mathrm{F}^{\alpha_-}_{\alpha_-,\alpha_+,\alpha_-} = \mathrm{F}^{\alpha_+}_{\alpha_-,\alpha_-,\alpha_-} = \begin{pmatrix} \frac{i}{\sqrt{3}} & 0 & \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} \\ 0 & -\frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} & 0 & -\frac{i}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} & 0 \end{pmatrix}$

- $\mathrm{F}^{\beta_2}_{\beta_2,\beta_2,\beta_2} = \mathrm{F}^{\beta_3}_{\beta_3,\beta_3,\beta_3} = \mathrm{F}^{\beta_4}_{\beta_4,\beta_4,\beta_4} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$

- $\mathrm{F}^{\beta_1}_{\beta_1,\beta_1,\beta_1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$

- $F^d_{abc} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

- $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_3}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_3,\alpha_-}$

- $F^d_{abc} = \begin{pmatrix} \frac{i\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{i\sqrt{3}}{2} \end{pmatrix}$

  - $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\beta_3}$

- $F^d_{abc} = \begin{pmatrix} -\frac{1}{2} & \frac{i\sqrt{3}}{2} \\ \frac{i\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

  - $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\beta_3}$

- $F^d_{abc} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

  - $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_1,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_2,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_2,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_1,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_3,\beta_4}$

- $F^d_{abc} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

  - $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_3,\beta_4}$

- $F^d_{abc} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$
  - $\mathrm{F}^{\beta_3}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_2,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_2,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_3,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_4,\beta_2}$
- $F^d_{abc} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$
  - $\mathrm{F}^{\beta_2}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_2,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_3,\alpha_+}$
- $F^d_{abc} = \begin{pmatrix} -\frac{i\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{i\sqrt{3}}{2} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\beta_1}$
- $F^d_{abc} = \begin{pmatrix} -\frac{1}{2} & -\frac{i\sqrt{3}}{2} \\ -\frac{i\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\beta_1}$

- $F^d_{abc} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$
  - $\mathrm{F}^{\beta_2}_{\beta_1,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_2,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_3,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_4,\beta_3}$
- $F^d_{abc} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_1,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_2,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_4,\alpha_+}$
- $F^d_{abc} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_2,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_4,\alpha_-}$
- $F^d_{abc} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\beta_4}$,

$\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_4,\alpha_+}$

- $F^d_{abc} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_4,\alpha_-}$
- $F^d_{abc} = \begin{pmatrix} -1 \end{pmatrix}$
  - $\mathrm{F}^{\beta_1}_{\epsilon,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_1}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\epsilon,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\epsilon,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\epsilon,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\epsilon,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\epsilon,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\epsilon,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\epsilon,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\epsilon,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_2,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\beta_2,\beta_2}$, $\mathrm{F}^{\epsilon}_{\epsilon,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_3,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\beta_3,\beta_3}$, $\mathrm{F}^{\epsilon}_{\epsilon,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_4,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\beta_4,\beta_4}$, $\mathrm{F}^{\epsilon}_{\epsilon,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_4,\beta_4}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\epsilon,\beta_1}$, $\mathrm{F}^{\epsilon}_{\beta_1,\epsilon,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_1,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_1,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_1,\epsilon,\beta_4}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_1,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_1,\epsilon}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_1,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_2,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_2,\beta_3}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_3,\epsilon}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_3,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_4,\epsilon}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\epsilon}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\epsilon}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\epsilon}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\epsilon}$, $\mathrm{F}^{\epsilon}_{\alpha_+,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\epsilon}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\epsilon}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\epsilon}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\epsilon}$, $\mathrm{F}^{\epsilon}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\epsilon}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\epsilon}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\epsilon}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\epsilon}$, $\mathrm{F}^{\epsilon}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\epsilon}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\epsilon}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\epsilon}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\epsilon}$, $\mathrm{F}^{\epsilon}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_2,\epsilon,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\beta_2,\epsilon,\beta_2}$, $\mathrm{F}^{\epsilon}_{\beta_2,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_2,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_2,\epsilon,\beta_4}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_1,\beta_3}$, $\mathrm{F}^{\epsilon}_{\beta_2,\alpha_+,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\epsilon}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_2,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_2,\beta_2}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_3,\beta_1}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_3,\epsilon,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_3,\epsilon,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\epsilon,\beta_3}$, $\mathrm{F}^{\epsilon}_{\beta_3,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_3,\epsilon,\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_1,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_1,\beta_2}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_1,\beta_4}$, $\mathrm{F}^{\epsilon}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\epsilon}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_2,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_3,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_4,\epsilon}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_4,\epsilon}$, $\mathrm{F}^{\beta_2}_{\beta_4,\epsilon,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_4,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_4,\epsilon,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\epsilon,\beta_4}$, $\mathrm{F}^{\epsilon}_{\beta_4,\epsilon,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_1,\epsilon}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_1,\epsilon}$,

$\mathrm{F}^{\epsilon}_{\beta_4,\beta_1,\beta_3}$, $\mathrm{F}^{\epsilon}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\epsilon}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_2,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_3,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_4,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_4,\beta_4}$

- $F^d_{abc} = \begin{pmatrix} 1 \end{pmatrix}$

  - $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\mathrm{I},\mathrm{I}}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\mathrm{I},\epsilon}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\mathrm{I},\beta_1}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\mathrm{I},\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\mathrm{I},\alpha_-}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\mathrm{I},\beta_2}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\mathrm{I},\beta_3}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\mathrm{I},\beta_4}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\epsilon,\mathrm{I}}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\epsilon,\epsilon}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\epsilon,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\epsilon,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\epsilon,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\epsilon,\beta_2}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\epsilon,\beta_3}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\epsilon,\beta_4}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_1,\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_1,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\beta_1,\beta_1}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\beta_1,\beta_1}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_1,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_1,\beta_2}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_1,\beta_2}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_1,\beta_3}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_1,\beta_3}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_-,\mathrm{I}}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_-,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_-,\beta_1}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\alpha_-,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_2,\mathrm{I}}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_2,\epsilon}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_2,\beta_1}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_2,\alpha_-}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\beta_2,\beta_2}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\beta_2,\beta_2}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_2,\beta_2}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_2,\beta_3}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_2,\beta_3}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_3,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_3,\epsilon}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_3,\beta_1}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_3,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_3,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\beta_3,\beta_3}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\beta_3,\beta_3}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_3,\beta_3}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_3,\beta_4}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_3,\beta_4}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_4,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_4,\epsilon}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_4,\beta_1}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_4,\beta_2}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_4,\beta_2}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_4,\beta_3}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_4,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\beta_4,\beta_4}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\beta_4,\beta_4}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_4,\beta_4}$, $\mathrm{F}^{\epsilon}_{\epsilon,\mathrm{I},\mathrm{I}}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\mathrm{I},\epsilon}$, $\mathrm{F}^{\beta_1}_{\epsilon,\mathrm{I},\beta_1}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\mathrm{I},\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\mathrm{I},\alpha_-}$, $\mathrm{F}^{\beta_2}_{\epsilon,\mathrm{I},\beta_2}$, $\mathrm{F}^{\beta_3}_{\epsilon,\mathrm{I},\beta_3}$, $\mathrm{F}^{\beta_4}_{\epsilon,\mathrm{I},\beta_4}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\epsilon,\mathrm{I}}$, $\mathrm{F}^{\epsilon}_{\epsilon,\epsilon,\epsilon}$, $\mathrm{F}^{\beta_1}_{\epsilon,\epsilon,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\epsilon,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\epsilon,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\epsilon,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_3}_{\epsilon,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_4}_{\epsilon,\epsilon,\beta_4}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_1,\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_1,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\beta_1,\beta_1}$, $\mathrm{F}^{\epsilon}_{\epsilon,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_1,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_1,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_+,\mathrm{I}}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_+,\beta_1}$, $\mathrm{F}^{\epsilon}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_-,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\epsilon,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_2,\mathrm{I}}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_2,\epsilon}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_3,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_3,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_4,\epsilon}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_1,\mathrm{I},\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\beta_1,\mathrm{I},\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\mathrm{I},\beta_1}$, $\mathrm{F}^{\epsilon}_{\beta_1,\mathrm{I},\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_1,\mathrm{I},\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\mathrm{I},\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\mathrm{I},\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\mathrm{I},\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\mathrm{I},\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_1,\mathrm{I},\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_1,\mathrm{I},\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_1,\mathrm{I},\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_1,\mathrm{I},\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_1,\mathrm{I},\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_1,\mathrm{I},\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_1,\epsilon,\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\beta_1,\epsilon,\epsilon}$, $\mathrm{F}^{\beta_1}_{\beta_1,\epsilon,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\epsilon,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\epsilon,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\epsilon,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\epsilon,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_1,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_1,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_1,\epsilon,\beta_4}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_1,\mathrm{I}}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_1,\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_1,\mathrm{I}}$,

$$\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_1,\beta_1},\ \mathrm{F}^{\beta_3}_{\beta_1,\beta_1,\beta_2},\ \mathrm{F}^{\beta_4}_{\beta_1,\beta_1,\beta_2},\ \mathrm{F}^{\beta_2}_{\beta_1,\beta_1,\beta_3},\ \mathrm{F}^{\beta_4}_{\beta_1,\beta_1,\beta_3},\ \mathrm{F}^{\beta_2}_{\beta_1,\beta_1,\beta_4},\ \mathrm{F}^{\beta_3}_{\beta_1,\beta_1,\beta_4},\ \mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\mathrm{I}},$$

$$\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\epsilon},\ \mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\beta_1,\alpha_+,\alpha_+},\ \mathrm{F}^{\epsilon}_{\beta_1,\alpha_+,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\beta_1,\alpha_+,\alpha_-},\ \mathrm{F}^{\epsilon}_{\beta_1,\alpha_+,\alpha_-},\ \mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\epsilon},$$

$$\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\beta_1,\alpha_-,\alpha_+},\ \mathrm{F}^{\epsilon}_{\beta_1,\alpha_-,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\beta_1,\alpha_-,\alpha_-},\ \mathrm{F}^{\epsilon}_{\beta_1,\alpha_-,\alpha_-},\ \mathrm{F}^{\beta_3}_{\beta_1,\beta_2,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\beta_1,\beta_2,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\beta_1,\beta_2,\epsilon},\ \mathrm{F}^{\beta_3}_{\beta_1,\beta_2,\beta_1},$$

$$\mathrm{F}^{\beta_4}_{\beta_1,\beta_2,\beta_1},\ \mathrm{F}^{\beta_3}_{\beta_1,\beta_2,\beta_2},\ \mathrm{F}^{\beta_4}_{\beta_1,\beta_2,\beta_2},\ \mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_2,\beta_3},\ \mathrm{F}^{\beta_1}_{\beta_1,\beta_2,\beta_3},\ \mathrm{F}^{\beta_2}_{\beta_1,\beta_2,\beta_3},\ \mathrm{F}^{\beta_3}_{\beta_1,\beta_2,\beta_3},\ \mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_2,\beta_4},\ \mathrm{F}^{\beta_1}_{\beta_1,\beta_2,\beta_4},$$

$$\mathrm{F}^{\beta_2}_{\beta_1,\beta_2,\beta_4},\ \mathrm{F}^{\beta_4}_{\beta_1,\beta_2,\beta_4},\ \mathrm{F}^{\beta_2}_{\beta_1,\beta_3,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\beta_1,\beta_3,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\beta_1,\beta_3,\beta_1},\ \mathrm{F}^{\beta_4}_{\beta_1,\beta_3,\beta_1},\ \mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_3,\beta_2},\ \mathrm{F}^{\beta_1}_{\beta_1,\beta_3,\beta_2},\ \mathrm{F}^{\beta_2}_{\beta_1,\beta_3,\beta_2},$$

$$\mathrm{F}^{\beta_3}_{\beta_1,\beta_3,\beta_2},\ \mathrm{F}^{\beta_2}_{\beta_1,\beta_3,\beta_3},\ \mathrm{F}^{\beta_4}_{\beta_1,\beta_3,\beta_3},\ \mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_3,\beta_4},\ \mathrm{F}^{\epsilon}_{\beta_1,\beta_3,\beta_4},\ \mathrm{F}^{\beta_1}_{\beta_1,\beta_3,\beta_4},\ \mathrm{F}^{\beta_3}_{\beta_1,\beta_3,\beta_4},\ \mathrm{F}^{\beta_4}_{\beta_1,\beta_3,\beta_4},\ \mathrm{F}^{\beta_2}_{\beta_1,\beta_4,\mathrm{I}},$$

$$\mathrm{F}^{\beta_3}_{\beta_1,\beta_4,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\beta_1,\beta_4,\epsilon},\ \mathrm{F}^{\beta_2}_{\beta_1,\beta_4,\beta_1},\ \mathrm{F}^{\beta_3}_{\beta_1,\beta_4,\beta_1},\ \mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_4,\beta_2},\ \mathrm{F}^{\epsilon}_{\beta_1,\beta_4,\beta_2},\ \mathrm{F}^{\beta_1}_{\beta_1,\beta_4,\beta_2},\ \mathrm{F}^{\beta_2}_{\beta_1,\beta_4,\beta_2},\ \mathrm{F}^{\beta_4}_{\beta_1,\beta_4,\beta_2},$$

$$\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_4,\beta_3},\ \mathrm{F}^{\epsilon}_{\beta_1,\beta_4,\beta_3},\ \mathrm{F}^{\beta_1}_{\beta_1,\beta_4,\beta_3},\ \mathrm{F}^{\beta_3}_{\beta_1,\beta_4,\beta_3},\ \mathrm{F}^{\beta_4}_{\beta_1,\beta_4,\beta_3},\ \mathrm{F}^{\beta_2}_{\beta_1,\beta_4,\beta_4},\ \mathrm{F}^{\beta_3}_{\beta_1,\beta_4,\beta_4},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\mathrm{I},\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\mathrm{I},\epsilon},$$

$$\mathrm{F}^{\alpha_+}_{\alpha_+,\mathrm{I},\beta_1},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\mathrm{I},\beta_1},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\mathrm{I},\alpha_+},\ \mathrm{F}^{\beta_1}_{\alpha_+,\mathrm{I},\alpha_+},\ \mathrm{F}^{\beta_2}_{\alpha_+,\mathrm{I},\alpha_+},\ \mathrm{F}^{\beta_3}_{\alpha_+,\mathrm{I},\alpha_+},\ \mathrm{F}^{\beta_4}_{\alpha_+,\mathrm{I},\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_+,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_1}_{\alpha_+,\mathrm{I},\alpha_-},$$

$$\mathrm{F}^{\beta_2}_{\alpha_+,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_3}_{\alpha_+,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_4}_{\alpha_+,\mathrm{I},\alpha_-},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\mathrm{I},\beta_2},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\mathrm{I},\beta_2},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\mathrm{I},\beta_3},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\mathrm{I},\beta_3},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\mathrm{I},\beta_4},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\mathrm{I},\beta_4},$$

$$\mathrm{F}^{\alpha_-}_{\alpha_+,\epsilon,\mathrm{I}},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\epsilon,\epsilon},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\epsilon,\beta_1},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\epsilon,\beta_1},\ \mathrm{F}^{\epsilon}_{\alpha_+,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_1}_{\alpha_+,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_2}_{\alpha_+,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_3}_{\alpha_+,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_4}_{\alpha_+,\epsilon,\alpha_+},$$

$$\mathrm{F}^{\mathrm{I}}_{\alpha_+,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_1}_{\alpha_+,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_2}_{\alpha_+,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_3}_{\alpha_+,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_4}_{\alpha_+,\epsilon,\alpha_-},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\epsilon,\beta_2},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\epsilon,\beta_2},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\epsilon,\beta_3},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\epsilon,\beta_3},$$

$$\mathrm{F}^{\alpha_+}_{\alpha_+,\epsilon,\beta_4},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\epsilon,\beta_4},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\mathrm{I}},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\epsilon},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_1,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_+,\beta_1,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_1,\alpha_-},$$

$$\mathrm{F}^{\epsilon}_{\alpha_+,\beta_1,\alpha_-},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\epsilon}_{\alpha_+,\alpha_+,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_+,\beta_1},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_+,\beta_2},$$

$$\mathrm{F}^{\epsilon}_{\alpha_+,\alpha_+,\beta_2},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_+,\beta_3},\ \mathrm{F}^{\epsilon}_{\alpha_+,\alpha_+,\beta_3},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_+,\beta_4},\ \mathrm{F}^{\epsilon}_{\alpha_+,\alpha_+,\beta_4},\ \mathrm{F}^{\epsilon}_{\alpha_+,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\mathrm{I}},$$

$$\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_-,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_-,\beta_1},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_-,\beta_2},\ \mathrm{F}^{\epsilon}_{\alpha_+,\alpha_-,\beta_2},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_-,\beta_3},\ \mathrm{F}^{\epsilon}_{\alpha_+,\alpha_-,\beta_3},$$

$$\mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_-,\beta_4},\ \mathrm{F}^{\epsilon}_{\alpha_+,\alpha_-,\beta_4},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\mathrm{I}},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\epsilon},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_2,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_+,\beta_2,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_2,\alpha_-},$$

$$\mathrm{F}^{\epsilon}_{\alpha_+,\beta_2,\alpha_-},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\mathrm{I}},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\epsilon},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_3,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_+,\beta_3,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_3,\alpha_-},\ \mathrm{F}^{\epsilon}_{\alpha_+,\beta_3,\alpha_-},$$

$$\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\mathrm{I}},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\epsilon},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_4,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_+,\beta_4,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_4,\alpha_-},\ \mathrm{F}^{\epsilon}_{\alpha_+,\beta_4,\alpha_-},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\mathrm{I},\mathrm{I}},$$

$$\mathrm{F}^{\alpha_+}_{\alpha_-,\mathrm{I},\epsilon},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\mathrm{I},\beta_1},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\mathrm{I},\beta_1},\ \mathrm{F}^{\epsilon}_{\alpha_-,\mathrm{I},\alpha_+},\ \mathrm{F}^{\beta_1}_{\alpha_-,\mathrm{I},\alpha_+},\ \mathrm{F}^{\beta_2}_{\alpha_-,\mathrm{I},\alpha_+},\ \mathrm{F}^{\beta_3}_{\alpha_-,\mathrm{I},\alpha_+},\ \mathrm{F}^{\beta_4}_{\alpha_-,\mathrm{I},\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\mathrm{I},\alpha_-},$$

$$\mathrm{F}^{\beta_1}_{\alpha_-,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_2}_{\alpha_-,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_3}_{\alpha_-,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_4}_{\alpha_-,\mathrm{I},\alpha_-},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\mathrm{I},\beta_2},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\mathrm{I},\beta_2},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\mathrm{I},\beta_3},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\mathrm{I},\beta_3},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\mathrm{I},\beta_4},$$

$$\mathrm{F}^{\alpha_-}_{\alpha_-,\mathrm{I},\beta_4},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\epsilon,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\epsilon,\epsilon},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\epsilon,\beta_1},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\epsilon,\beta_1},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_1}_{\alpha_-,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_2}_{\alpha_-,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_3}_{\alpha_-,\epsilon,\alpha_+},$$

$$\mathrm{F}^{\beta_4}_{\alpha_-,\epsilon,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_-,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_1}_{\alpha_-,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_2}_{\alpha_-,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_3}_{\alpha_-,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_4}_{\alpha_-,\epsilon,\alpha_-},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\epsilon,\beta_2},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\epsilon,\beta_2},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\epsilon,\beta_3},$$

$$\mathrm{F}^{\alpha_-}_{\alpha_-,\epsilon,\beta_3},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\epsilon,\beta_4},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\epsilon,\beta_4},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\mathrm{I}},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\epsilon},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_1,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_1,\alpha_+},$$

$$\mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_1,\alpha_-},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_1,\alpha_-},\ \mathrm{F}^{\epsilon}_{\alpha_-,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_+,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_+,\beta_1},$$

$$\mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_+,\beta_2},\ \mathrm{F}^{\epsilon}_{\alpha_-,\alpha_+,\beta_2},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_+,\beta_3},\ \mathrm{F}^{\epsilon}_{\alpha_-,\alpha_+,\beta_3},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_+,\beta_4},\ \mathrm{F}^{\epsilon}_{\alpha_-,\alpha_+,\beta_4},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\mathrm{I}},$$

$$\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\epsilon}_{\alpha_-,\alpha_-,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_-,\beta_1},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_-,\beta_2},\ \mathrm{F}^{\epsilon}_{\alpha_-,\alpha_-,\beta_2},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_-,\beta_3},\ \mathrm{F}^{\epsilon}_{\alpha_-,\alpha_-,\beta_3},$$

$$\mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_-,\beta_4},\ \mathrm{F}^{\epsilon}_{\alpha_-,\alpha_-,\beta_4},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\mathrm{I}},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\epsilon},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_2,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_2,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_2,\alpha_-},$$

$$\mathrm{F}^{\epsilon}_{\alpha_-,\beta_2,\alpha_-},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\mathrm{I}},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\epsilon},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_3,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_3,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_3,\alpha_-},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_3,\alpha_-},$$

$$\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\mathrm{I}},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\epsilon},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_4,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_4,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_4,\alpha_-},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_4,\alpha_-},\ \mathrm{F}^{\beta_2}_{\beta_2,\mathrm{I},\mathrm{I}},$$

$$\mathrm{F}^{\beta_2}_{\beta_2,\mathrm{I},\epsilon},\ \mathrm{F}^{\beta_3}_{\beta_2,\mathrm{I},\beta_1},\ \mathrm{F}^{\beta_4}_{\beta_2,\mathrm{I},\beta_1},\ \mathrm{F}^{\alpha_+}_{\beta_2,\mathrm{I},\alpha_+},\ \mathrm{F}^{\alpha_-}_{\beta_2,\mathrm{I},\alpha_+},\ \mathrm{F}^{\alpha_+}_{\beta_2,\mathrm{I},\alpha_-},\ \mathrm{F}^{\alpha_-}_{\beta_2,\mathrm{I},\alpha_-},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\mathrm{I},\beta_2},\ \mathrm{F}^{\epsilon}_{\beta_2,\mathrm{I},\beta_2},\ \mathrm{F}^{\beta_2}_{\beta_2,\mathrm{I},\beta_2},$$

$$\mathrm{F}^{\beta_1}_{\beta_2,\mathrm{I},\beta_3},\ \mathrm{F}^{\beta_4}_{\beta_2,\mathrm{I},\beta_3},\ \mathrm{F}^{\beta_1}_{\beta_2,\mathrm{I},\beta_4},\ \mathrm{F}^{\beta_3}_{\beta_2,\mathrm{I},\beta_4},\ \mathrm{F}^{\beta_2}_{\beta_2,\epsilon,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\beta_2,\epsilon,\epsilon},\ \mathrm{F}^{\beta_4}_{\beta_2,\epsilon,\beta_1},\ \mathrm{F}^{\alpha_+}_{\beta_2,\epsilon,\alpha_+},\ \mathrm{F}^{\alpha_-}_{\beta_2,\epsilon,\alpha_+},\ \mathrm{F}^{\alpha_+}_{\beta_2,\epsilon,\alpha_-},$$

$$\mathrm{F}^{\alpha_-}_{\beta_2,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_2}_{\beta_2,\epsilon,\beta_2},\ \mathrm{F}^{\beta_1}_{\beta_2,\epsilon,\beta_3},\ \mathrm{F}^{\beta_3}_{\beta_2,\epsilon,\beta_4},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_1,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_1,\mathrm{I}},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_1,\epsilon},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_1,\epsilon},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_1,\beta_1},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_1,\beta_1},$$

$$\mathrm{F}^{\beta_3}_{\beta_2,\beta_1,\beta_2},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_1,\beta_2},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_1,\beta_3},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_1,\beta_3},\ \mathrm{F}^{\beta_2}_{\beta_2,\beta_1,\beta_3},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_1,\beta_3},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_1,\beta_4},\ \mathrm{F}^{\epsilon}_{\beta_2,\beta_1,\beta_4},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_1,\beta_4},$$

$$\mathrm{F}^{\beta_2}_{\beta_2,\beta_1,\beta_4},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_1,\beta_4},\ \mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\epsilon},\ \mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\alpha_+,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\alpha_+,\alpha_-},\ \mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\mathrm{I}},$$

$$\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\epsilon},\ \mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\alpha_-,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\alpha_-,\alpha_-},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_2,\mathrm{I}},\ \mathrm{F}^{\epsilon}_{\beta_2,\beta_2,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\beta_2,\beta_2,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_2,\epsilon},$$

$$\mathrm{F}^{\epsilon}_{\beta_2,\beta_2,\epsilon},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_2,\beta_1},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_2,\beta_1},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_2,\beta_2},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_2,\beta_3},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_2,\beta_3},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_2,\beta_4},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_2,\beta_4},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_3,\mathrm{I}},$$

$$\mathrm{F}^{\beta_4}_{\beta_2,\beta_3,\mathrm{I}},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_3,\epsilon},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_3,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_3,\beta_1},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_3,\beta_1},\ \mathrm{F}^{\beta_2}_{\beta_2,\beta_3,\beta_1},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_3,\beta_1},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_3,\beta_2},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_3,\beta_2},$$

$$\mathrm{F}^{\beta_1}_{\beta_2,\beta_3,\beta_3},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_3,\beta_3},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_3,\beta_4},\ \mathrm{F}^{\beta_2}_{\beta_2,\beta_3,\beta_4},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_3,\beta_4},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_3,\beta_4},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_4,\mathrm{I}},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_4,\mathrm{I}},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_4,\epsilon},$$

$$\mathrm{F}^{\beta_3}_{\beta_2,\beta_4,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_4,\beta_1},\ \mathrm{F}^{\epsilon}_{\beta_2,\beta_4,\beta_1},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_4,\beta_1},\ \mathrm{F}^{\beta_2}_{\beta_2,\beta_4,\beta_1},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_4,\beta_1},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_4,\beta_2},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_4,\beta_2},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_4,\beta_3},$$

$$\mathrm{F}^{\epsilon}_{\beta_2,\beta_4,\beta_3},\ \mathrm{F}^{\beta_2}_{\beta_2,\beta_4,\beta_3},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_4,\beta_3},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_4,\beta_3},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_4,\beta_4},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_4,\beta_4},\ \mathrm{F}^{\beta_3}_{\beta_3,\mathrm{I},\mathrm{I}},\ \mathrm{F}^{\beta_3}_{\beta_3,\mathrm{I},\epsilon},\ \mathrm{F}^{\beta_2}_{\beta_3,\mathrm{I},\beta_1},\ \mathrm{F}^{\beta_4}_{\beta_3,\mathrm{I},\beta_1},$$

$$\mathrm{F}^{\alpha_+}_{\beta_3,\mathrm{I},\alpha_+},\ \mathrm{F}^{\alpha_-}_{\beta_3,\mathrm{I},\alpha_+},\ \mathrm{F}^{\alpha_+}_{\beta_3,\mathrm{I},\alpha_-},\ \mathrm{F}^{\alpha_-}_{\beta_3,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_1}_{\beta_3,\mathrm{I},\beta_2},\ \mathrm{F}^{\beta_4}_{\beta_3,\mathrm{I},\beta_2},\ \mathrm{F}^{\mathrm{I}}_{\beta_3,\mathrm{I},\beta_3},\ \mathrm{F}^{\epsilon}_{\beta_3,\mathrm{I},\beta_3},\ \mathrm{F}^{\beta_3}_{\beta_3,\mathrm{I},\beta_3},\ \mathrm{F}^{\beta_1}_{\beta_3,\mathrm{I},\beta_4},$$

$$\mathrm{F}^{\beta_2}_{\beta_3,\mathrm{I},\beta_4},\ \mathrm{F}^{\beta_3}_{\beta_3,\epsilon,\mathrm{I}},\ \mathrm{F}^{\beta_3}_{\beta_3,\epsilon,\epsilon},\ \mathrm{F}^{\beta_4}_{\beta_3,\epsilon,\beta_1},\ \mathrm{F}^{\alpha_+}_{\beta_3,\epsilon,\alpha_+},\ \mathrm{F}^{\alpha_-}_{\beta_3,\epsilon,\alpha_+},\ \mathrm{F}^{\alpha_+}_{\beta_3,\epsilon,\alpha_-},\ \mathrm{F}^{\alpha_-}_{\beta_3,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_1}_{\beta_3,\epsilon,\beta_2},\ \mathrm{F}^{\beta_3}_{\beta_3,\epsilon,\beta_3},$$

$$\mathrm{F}^{\beta_1}_{\beta_3,\epsilon,\beta_4},\ \mathrm{F}^{\beta_2}_{\beta_3,\beta_1,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\beta_3,\beta_1,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\beta_3,\beta_1,\epsilon},\ \mathrm{F}^{\beta_2}_{\beta_3,\beta_1,\beta_1},\ \mathrm{F}^{\beta_4}_{\beta_3,\beta_1,\beta_1},\ \mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_1,\beta_2},\ \mathrm{F}^{\beta_1}_{\beta_3,\beta_1,\beta_2},\ \mathrm{F}^{\beta_2}_{\beta_3,\beta_1,\beta_2},$$

$$\mathrm{F}^{\beta_3}_{\beta_3,\beta_1,\beta_2},\ \mathrm{F}^{\beta_2}_{\beta_3,\beta_1,\beta_3},\ \mathrm{F}^{\beta_4}_{\beta_3,\beta_1,\beta_3},\ \mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_1,\beta_4},\ \mathrm{F}^{\beta_1}_{\beta_3,\beta_1,\beta_4},\ \mathrm{F}^{\beta_3}_{\beta_3,\beta_1,\beta_4},\ \mathrm{F}^{\beta_4}_{\beta_3,\beta_1,\beta_4},\ \mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\mathrm{I}},$$

$$\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\epsilon},\ \mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\beta_3,\alpha_+,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\beta_3,\alpha_+,\alpha_-},\ \mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\epsilon},\ \mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\beta_3,\alpha_-,\alpha_+},$$

$$\mathrm{F}^{\mathrm{I}}_{\beta_3,\alpha_-,\alpha_-},\ \mathrm{F}^{\beta_1}_{\beta_3,\beta_2,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\beta_3,\beta_2,\mathrm{I}},\ \mathrm{F}^{\beta_1}_{\beta_3,\beta_2,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_2,\beta_1},\ \mathrm{F}^{\beta_1}_{\beta_3,\beta_2,\beta_1},\ \mathrm{F}^{\beta_2}_{\beta_3,\beta_2,\beta_1},\ \mathrm{F}^{\beta_3}_{\beta_3,\beta_2,\beta_1},\ \mathrm{F}^{\beta_1}_{\beta_3,\beta_2,\beta_2},$$

$$\mathrm{F}^{\beta_4}_{\beta_3,\beta_2,\beta_2},\ \mathrm{F}^{\beta_1}_{\beta_3,\beta_2,\beta_3},\ \mathrm{F}^{\beta_4}_{\beta_3,\beta_2,\beta_3},\ \mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_2,\beta_4},\ \mathrm{F}^{\epsilon}_{\beta_3,\beta_2,\beta_4},\ \mathrm{F}^{\beta_2}_{\beta_3,\beta_2,\beta_4},\ \mathrm{F}^{\beta_3}_{\beta_3,\beta_2,\beta_4},\ \mathrm{F}^{\beta_4}_{\beta_3,\beta_2,\beta_4},\ \mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_3,\mathrm{I}},$$

$$\mathrm{F}^{\epsilon}_{\beta_3,\beta_3,\mathrm{I}},\ \mathrm{F}^{\beta_3}_{\beta_3,\beta_3,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_3,\epsilon},\ \mathrm{F}^{\epsilon}_{\beta_3,\beta_3,\epsilon},\ \mathrm{F}^{\beta_2}_{\beta_3,\beta_3,\beta_1},\ \mathrm{F}^{\beta_4}_{\beta_3,\beta_3,\beta_1},\ \mathrm{F}^{\beta_1}_{\beta_3,\beta_3,\beta_2},\ \mathrm{F}^{\beta_4}_{\beta_3,\beta_3,\beta_2},\ \mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_3,\beta_3},$$

$$\mathrm{F}^{\beta_1}_{\beta_3,\beta_3,\beta_4},\ \mathrm{F}^{\beta_2}_{\beta_3,\beta_3,\beta_4},\ \mathrm{F}^{\beta_1}_{\beta_3,\beta_4,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\beta_3,\beta_4,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_4,\beta_1},\ \mathrm{F}^{\epsilon}_{\beta_3,\beta_4,\beta_1},\ \mathrm{F}^{\beta_1}_{\beta_3,\beta_4,\beta_1},\ \mathrm{F}^{\beta_3}_{\beta_3,\beta_4,\beta_1},\ \mathrm{F}^{\beta_4}_{\beta_3,\beta_4,\beta_1},$$

$$\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_4,\beta_2},\ \mathrm{F}^{\epsilon}_{\beta_3,\beta_4,\beta_2},\ \mathrm{F}^{\beta_2}_{\beta_3,\beta_4,\beta_2},\ \mathrm{F}^{\beta_3}_{\beta_3,\beta_4,\beta_2},\ \mathrm{F}^{\beta_4}_{\beta_3,\beta_4,\beta_2},\ \mathrm{F}^{\beta_1}_{\beta_3,\beta_4,\beta_3},\ \mathrm{F}^{\beta_2}_{\beta_3,\beta_4,\beta_3},\ \mathrm{F}^{\beta_1}_{\beta_3,\beta_4,\beta_4},\ \mathrm{F}^{\beta_2}_{\beta_3,\beta_4,\beta_4},$$

$$\mathrm{F}^{\beta_4}_{\beta_4,\mathrm{I},\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\beta_4,\mathrm{I},\epsilon},\ \mathrm{F}^{\beta_2}_{\beta_4,\mathrm{I},\beta_1},\ \mathrm{F}^{\beta_3}_{\beta_4,\mathrm{I},\beta_1},\ \mathrm{F}^{\alpha_+}_{\beta_4,\mathrm{I},\alpha_+},\ \mathrm{F}^{\alpha_-}_{\beta_4,\mathrm{I},\alpha_+},\ \mathrm{F}^{\alpha_+}_{\beta_4,\mathrm{I},\alpha_-},\ \mathrm{F}^{\alpha_-}_{\beta_4,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_1}_{\beta_4,\mathrm{I},\beta_2},\ \mathrm{F}^{\beta_3}_{\beta_4,\mathrm{I},\beta_2},$$

$\mathrm{F}^{\beta_1}_{\beta_4,\mathrm{I},\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_4,\mathrm{I},\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\mathrm{I},\beta_4}$, $\mathrm{F}^{\epsilon}_{\beta_4,\mathrm{I},\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_4,\mathrm{I},\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_4,\epsilon,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\beta_4,\epsilon,\epsilon}$, $\mathrm{F}^{\beta_3}_{\beta_4,\epsilon,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\epsilon,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\epsilon,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\epsilon,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\epsilon,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_4,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_4,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_4,\epsilon,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_1,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_1,\mathrm{I}}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_1,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_1,\beta_2}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_1,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\mathrm{I}}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\mathrm{I}}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_2,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_2,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_2,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_2,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_2,\beta_3}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_3,\mathrm{I}}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_3,\mathrm{I}}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_3,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_3,\beta_1}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_3,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_3,\beta_4}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_4,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_4,\epsilon}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_4,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_4,\beta_4}$

A.1.2 Pivotal Structures

There are 2 pivotal structures.

- $\epsilon_{\mathrm{I}} = 1$, $\epsilon_{\epsilon} = 1$, $\epsilon_{\beta_1} = 1$, $\epsilon_{\alpha_+} = 1$, $\epsilon_{\alpha_-} = 1$, $\epsilon_{\beta_2} = 1$, $\epsilon_{\beta_3} = 1$, and $\epsilon_{\beta_4} = 1$ which is spherical with quantum dimensions $q_{\mathrm{I}} = 1$, $q_{\epsilon} = 1$, $q_{\beta_1} = 2$, $q_{\alpha_+} = 3$, $q_{\alpha_-} = 3$, $q_{\beta_2} = 2$, $q_{\beta_3} = 2$, and $q_{\beta_4} = 2$.

- $\epsilon_{\mathrm{I}} = 1$, $\epsilon_{\epsilon} = 1$, $\epsilon_{\beta_1} = 1$, $\epsilon_{\alpha_+} = -1$, $\epsilon_{\alpha_-} = -1$, $\epsilon_{\beta_2} = 1$, $\epsilon_{\beta_3} = 1$, and $\epsilon_{\beta_4} = 1$ which is spherical with quantum dimensions $q_{\mathrm{I}} = 1$, $q_{\epsilon} = 1$, $q_{\beta_1} = 2$, $q_{\alpha_+} = -3$, $q_{\alpha_-} = -3$, $q_{\beta_2} = 2$, $q_{\beta_3} = 2$, and $q_{\beta_4} = 2$.

A.1.3 Braidings

There is 1 braidings which correspond to this solution. it is not symmetric.

- $\mathrm{R}^{\alpha_-}_{\beta_1,\alpha_+} = \mathrm{R}^{\alpha_+}_{\beta_1,\alpha_-} = \mathrm{R}^{\beta_2}_{\alpha_-,\alpha_+} = \mathrm{R}^{\beta_2}_{\alpha_-,\alpha_-} = \mathrm{R}^{\epsilon}_{\beta_2,\beta_2} = \begin{pmatrix} \frac{1}{2}\left(1 + i\sqrt{3}\right) \end{pmatrix}$
- $R^{c}_{a,b} = \begin{pmatrix} -\frac{1}{2}i\left(-i + \sqrt{3}\right) \end{pmatrix}$

  - $\mathrm{R}^{\beta_1}_{\beta_1,\beta_1}$, $\mathrm{R}^{\alpha_+}_{\beta_1,\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\beta_1,\alpha_-}$, $\mathrm{R}^{\beta_4}_{\beta_1,\beta_3}$, $\mathrm{R}^{\beta_3}_{\beta_1,\beta_4}$, $\mathrm{R}^{\alpha_+}_{\alpha_+,\beta_1}$, $\mathrm{R}^{\alpha_-}_{\alpha_+,\beta_1}$, $\mathrm{R}^{\beta_2}_{\alpha_+,\alpha_+}$, $\mathrm{R}^{\beta_2}_{\alpha_+,\alpha_-}$, $\mathrm{R}^{\alpha_+}_{\alpha_-,\beta_1}$, $\mathrm{R}^{\alpha_-}_{\alpha_-,\beta_1}$, $\mathrm{R}^{\mathrm{I}}_{\beta_2,\beta_2}$, $\mathrm{R}^{\beta_1}_{\beta_2,\beta_3}$, $\mathrm{R}^{\beta_1}_{\beta_2,\beta_4}$, $\mathrm{R}^{\beta_4}_{\beta_3,\beta_1}$, $\mathrm{R}^{\beta_1}_{\beta_3,\beta_2}$, $\mathrm{R}^{\beta_2}_{\beta_3,\beta_4}$, $\mathrm{R}^{\beta_3}_{\beta_4,\beta_1}$, $\mathrm{R}^{\beta_1}_{\beta_4,\beta_2}$, $\mathrm{R}^{\beta_2}_{\beta_4,\beta_3}$

- $\mathrm{R}^{\epsilon}_{\beta_1,\beta_1} = \mathrm{R}^{\beta_1}_{\alpha_-,\alpha_+} = \mathrm{R}^{\beta_1}_{\alpha_-,\alpha_-} = \mathrm{R}^{\alpha_-}_{\beta_2,\alpha_+} = \mathrm{R}^{\alpha_+}_{\beta_2,\alpha_-} = \begin{pmatrix} \frac{1}{2}\left(1 - i\sqrt{3}\right) \end{pmatrix}$

- $R^c_{a,b} = \begin{pmatrix} \frac{1}{2}i\left(i + \sqrt{3}\right) \end{pmatrix}$

  - $\mathrm{R}^{\mathrm{I}}_{\beta_1,\beta_1}$, $\mathrm{R}^{\beta_2}_{\beta_1,\beta_3}$, $\mathrm{R}^{\beta_2}_{\beta_1,\beta_4}$, $\mathrm{R}^{\beta_1}_{\alpha_+,\alpha_+}$, $\mathrm{R}^{\beta_1}_{\alpha_+,\alpha_-}$, $\mathrm{R}^{\alpha_+}_{\alpha_+,\beta_2}$, $\mathrm{R}^{\alpha_-}_{\alpha_+,\beta_2}$, $\mathrm{R}^{\alpha_+}_{\alpha_-,\beta_2}$, $\mathrm{R}^{\alpha_-}_{\alpha_-,\beta_2}$, $\mathrm{R}^{\alpha_+}_{\beta_2,\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\beta_2,\alpha_-}$, $\mathrm{R}^{\beta_2}_{\beta_2,\beta_2}$, $\mathrm{R}^{\beta_4}_{\beta_2,\beta_3}$, $\mathrm{R}^{\beta_3}_{\beta_2,\beta_4}$, $\mathrm{R}^{\beta_2}_{\beta_3,\beta_1}$, $\mathrm{R}^{\beta_4}_{\beta_3,\beta_2}$, $\mathrm{R}^{\beta_1}_{\beta_3,\beta_4}$, $\mathrm{R}^{\beta_2}_{\beta_4,\beta_1}$, $\mathrm{R}^{\beta_3}_{\beta_4,\beta_2}$, $\mathrm{R}^{\beta_1}_{\beta_4,\beta_3}$

- $R^c_{a,b} = \begin{pmatrix} -1 \end{pmatrix}$

  - $\mathrm{R}^{\alpha_-}_{\epsilon,\alpha_+}$, $\mathrm{R}^{\alpha_+}_{\epsilon,\alpha_-}$, $\mathrm{R}^{\epsilon}_{\alpha_-,\alpha_+}$, $\mathrm{R}^{\beta_3}_{\alpha_-,\alpha_+}$, $\mathrm{R}^{\beta_4}_{\alpha_-,\alpha_+}$, $\mathrm{R}^{\mathrm{I}}_{\alpha_-,\alpha_-}$, $\mathrm{R}^{\beta_3}_{\alpha_-,\alpha_-}$, $\mathrm{R}^{\beta_4}_{\alpha_-,\alpha_-}$, $\mathrm{R}^{\alpha_-}_{\beta_3,\alpha_+}$, $\mathrm{R}^{\alpha_+}_{\beta_3,\alpha_-}$, $\mathrm{R}^{\epsilon}_{\beta_3,\beta_3}$, $\mathrm{R}^{\alpha_-}_{\beta_4,\alpha_+}$, $\mathrm{R}^{\alpha_+}_{\beta_4,\alpha_-}$, $\mathrm{R}^{\epsilon}_{\beta_4,\beta_4}$

- $R^c_{a,b} = \begin{pmatrix} 1 \end{pmatrix}$

  - $\mathrm{R}^{\mathrm{I}}_{\mathrm{I},\mathrm{I}}$, $\mathrm{R}^{\epsilon}_{\mathrm{I},\epsilon}$, $\mathrm{R}^{\beta_1}_{\mathrm{I},\beta_1}$, $\mathrm{R}^{\alpha_+}_{\mathrm{I},\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\mathrm{I},\alpha_-}$, $\mathrm{R}^{\beta_2}_{\mathrm{I},\beta_2}$, $\mathrm{R}^{\beta_3}_{\mathrm{I},\beta_3}$, $\mathrm{R}^{\beta_4}_{\mathrm{I},\beta_4}$, $\mathrm{R}^{\epsilon}_{\epsilon,\mathrm{I}}$, $\mathrm{R}^{\mathrm{I}}_{\epsilon,\epsilon}$, $\mathrm{R}^{\beta_1}_{\epsilon,\beta_1}$, $\mathrm{R}^{\beta_2}_{\epsilon,\beta_2}$, $\mathrm{R}^{\beta_3}_{\epsilon,\beta_3}$, $\mathrm{R}^{\beta_4}_{\epsilon,\beta_4}$, $\mathrm{R}^{\beta_1}_{\beta_1,\mathrm{I}}$, $\mathrm{R}^{\beta_1}_{\beta_1,\epsilon}$, $\mathrm{R}^{\beta_3}_{\beta_1,\beta_2}$, $\mathrm{R}^{\beta_4}_{\beta_1,\beta_2}$, $\mathrm{R}^{\alpha_+}_{\alpha_+,\mathrm{I}}$, $\mathrm{R}^{\alpha_-}_{\alpha_+,\epsilon}$, $\mathrm{R}^{\mathrm{I}}_{\alpha_+,\alpha_+}$, $\mathrm{R}^{\beta_3}_{\alpha_+,\alpha_+}$, $\mathrm{R}^{\beta_4}_{\alpha_+,\alpha_+}$, $\mathrm{R}^{\epsilon}_{\alpha_+,\alpha_-}$, $\mathrm{R}^{\beta_3}_{\alpha_+,\alpha_-}$, $\mathrm{R}^{\beta_4}_{\alpha_+,\alpha_-}$, $\mathrm{R}^{\alpha_+}_{\alpha_+,\beta_3}$, $\mathrm{R}^{\alpha_-}_{\alpha_+,\beta_3}$, $\mathrm{R}^{\alpha_+}_{\alpha_+,\beta_4}$, $\mathrm{R}^{\alpha_-}_{\alpha_+,\beta_4}$, $\mathrm{R}^{\alpha_-}_{\alpha_-,\mathrm{I}}$, $\mathrm{R}^{\alpha_+}_{\alpha_-,\epsilon}$, $\mathrm{R}^{\alpha_+}_{\alpha_-,\beta_3}$, $\mathrm{R}^{\alpha_-}_{\alpha_-,\beta_3}$, $\mathrm{R}^{\alpha_+}_{\alpha_-,\beta_4}$, $\mathrm{R}^{\alpha_-}_{\alpha_-,\beta_4}$, $\mathrm{R}^{\beta_2}_{\beta_2,\mathrm{I}}$, $\mathrm{R}^{\beta_2}_{\beta_2,\epsilon}$, $\mathrm{R}^{\beta_3}_{\beta_2,\beta_1}$, $\mathrm{R}^{\beta_4}_{\beta_2,\beta_1}$, $\mathrm{R}^{\beta_3}_{\beta_3,\mathrm{I}}$, $\mathrm{R}^{\beta_3}_{\beta_3,\epsilon}$, $\mathrm{R}^{\alpha_+}_{\beta_3,\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\beta_3,\alpha_-}$, $\mathrm{R}^{\mathrm{I}}_{\beta_3,\beta_3}$, $\mathrm{R}^{\beta_3}_{\beta_3,\beta_3}$, $\mathrm{R}^{\beta_4}_{\beta_4,\mathrm{I}}$, $\mathrm{R}^{\beta_4}_{\beta_4,\epsilon}$, $\mathrm{R}^{\alpha_+}_{\beta_4,\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\beta_4,\alpha_-}$, $\mathrm{R}^{\mathrm{I}}_{\beta_4,\beta_4}$, $\mathrm{R}^{\beta_4}_{\beta_4,\beta_4}$

A.1.4 Balanced, Ribbon, and Modular Structures

There are 2 balanced categories which correspond to this solution. Both of them are modular.

Balanced Category 1 has pivotal structure 1 and braiding 1. Its T-Matrix is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is ribbon and has S-Matrix given by:

$$\begin{pmatrix} 1 & 1 & 2 & 3 & 3 & 2 & 2 & 2 \\ 1 & 1 & 2 & -3 & -3 & 2 & 2 & 2 \\ 2 & 2 & -2 & 0 & 0 & 4 & -2 & -2 \\ 3 & -3 & 0 & 3 & -3 & 0 & 0 & 0 \\ 3 & -3 & 0 & -3 & 3 & 0 & 0 & 0 \\ 2 & 2 & 4 & 0 & 0 & -2 & -2 & -2 \\ 2 & 2 & -2 & 0 & 0 & -2 & 4 & -2 \\ 2 & 2 & -2 & 0 & 0 & -2 & -2 & 4 \end{pmatrix}.$$

Balanced Category 2 has pivotal structure 2 and braiding 1. Its T-Matrix is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is ribbon and has S-Matrix given by:

$$\begin{pmatrix} 1 & 1 & 2 & -3 & -3 & 2 & 2 & 2 \\ 1 & 1 & 2 & 3 & 3 & 2 & 2 & 2 \\ 2 & 2 & -2 & 0 & 0 & 4 & -2 & -2 \\ -3 & 3 & 0 & 3 & -3 & 0 & 0 & 0 \\ -3 & 3 & 0 & -3 & 3 & 0 & 0 & 0 \\ 2 & 2 & 4 & 0 & 0 & -2 & -2 & -2 \\ 2 & 2 & -2 & 0 & 0 & -2 & 4 & -2 \\ 2 & 2 & -2 & 0 & 0 & -2 & -2 & 4 \end{pmatrix}.$$

A.2 Solution 2 – $Rep(D^{\omega}(S_3))$

A.2.1 F-Matrices

- $\mathrm{F}^{\alpha_+}_{\alpha_+,\alpha_-,\alpha_-} = \mathrm{F}^{\alpha_-}_{\alpha_-,\alpha_+,\alpha_+} = \begin{pmatrix} -\frac{1}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{\sqrt{2}}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{\sqrt{2}}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

- $\mathrm{F}^{\alpha_-}_{\alpha_+,\alpha_-,\alpha_+} = \mathrm{F}^{\alpha_+}_{\alpha_-,\alpha_+,\alpha_-} = \begin{pmatrix} -\frac{1}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$

- $\mathrm{F}^{\alpha_-}_{\alpha_+,\alpha_+,\alpha_-} = \mathrm{F}^{\alpha_+}_{\alpha_-,\alpha_-,\alpha_+} = \begin{pmatrix} -\frac{1}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{\sqrt{2}}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

- $\mathrm{F}^{\alpha_+}_{\alpha_+,\alpha_+,\alpha_+} = \mathrm{F}^{\alpha_-}_{\alpha_-,\alpha_-,\alpha_-} = \begin{pmatrix} -\frac{1}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{\sqrt{2}}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$

- $\mathrm{F}^{\alpha_+}_{\alpha_+,\alpha_+,\alpha_-} = \mathrm{F}^{\alpha_-}_{\alpha_+,\alpha_-,\alpha_-} = \mathrm{F}^{\alpha_+}_{\alpha_-,\alpha_+,\alpha_+} = \mathrm{F}^{\alpha_-}_{\alpha_-,\alpha_-,\alpha_+} = \begin{pmatrix} -\frac{i}{\sqrt{3}} & 0 & -\frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \\ 0 & \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} \\ -\frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} & 0 & \frac{i}{\sqrt{3}} \\ -\frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} & 0 \end{pmatrix}$
- $\mathrm{F}^{\alpha_-}_{\alpha_+,\alpha_+,\alpha_+} = \mathrm{F}^{\alpha_+}_{\alpha_+,\alpha_-,\alpha_+} = \mathrm{F}^{\alpha_-}_{\alpha_-,\alpha_+,\alpha_-} = \mathrm{F}^{\alpha_+}_{\alpha_-,\alpha_-,\alpha_-} = \begin{pmatrix} \frac{i}{\sqrt{3}} & 0 & \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} \\ 0 & -\frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} & \frac{i}{\sqrt{3}} & 0 & -\frac{i}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} & -\frac{i}{\sqrt{3}} & 0 \end{pmatrix}$
- $\mathrm{F}^{\beta_2}_{\beta_2,\beta_2,\beta_2} = \mathrm{F}^{\beta_3}_{\beta_3,\beta_3,\beta_3} = \mathrm{F}^{\beta_4}_{\beta_4,\beta_4,\beta_4} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$
- $\mathrm{F}^{\beta_1}_{\beta_1,\beta_1,\beta_1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$
- $F^d_{abc} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$
  - $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_3}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_3,\alpha_-}$
- $F^d_{abc} = \begin{pmatrix} -\frac{i\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{i\sqrt{3}}{2} \end{pmatrix}$

– $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\beta_3}$

- $F^d_{abc} = \begin{pmatrix} -\frac{1}{2} & -\frac{i\sqrt{3}}{2} \\ -\frac{i\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$

– $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\beta_3}$

- $F^d_{abc} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

– $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_1,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_2,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_2,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_1,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_3,\beta_4}$

- $F^d_{abc} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

– $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\beta_4}$

- $F^d_{abc} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

– $\mathrm{F}^{\beta_3}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_2,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_2,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_3,\alpha_-,\alpha_+}$,

$\mathrm{F}^{\beta_1}_{\beta_3,\beta_3,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_4,\beta_2}$

- $F^d_{abc} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$
  - $\mathrm{F}^{\beta_2}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_2,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_3,\alpha_+}$
- $F^d_{abc} = \begin{pmatrix} \frac{i\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{i\sqrt{3}}{2} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\beta_1}$
- $F^d_{abc} = \begin{pmatrix} -\frac{1}{2} & \frac{i\sqrt{3}}{2} \\ \frac{i\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\beta_1}$
- $F^d_{abc} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$
  - $\mathrm{F}^{\beta_2}_{\beta_1,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_2,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_3,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_4,\beta_3}$

- $F^d_{abc} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_1,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_2,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_4,\alpha_+}$
- $F^d_{abc} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_2,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_4,\alpha_-}$
- $F^d_{abc} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_4,\alpha_+}$
- $F^d_{abc} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

- $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_4,\alpha_-}$

- $F^d_{abc} = \begin{pmatrix} -1 \end{pmatrix}$

  - $\mathrm{F}^{\beta_1}_{\epsilon,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_1}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\epsilon,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\epsilon,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\epsilon,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\epsilon,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\epsilon,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\epsilon,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\epsilon,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\epsilon,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_2,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\beta_2,\beta_2}$, $\mathrm{F}^{\epsilon}_{\epsilon,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_3,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\beta_3,\beta_3}$, $\mathrm{F}^{\epsilon}_{\epsilon,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_4,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\beta_4,\beta_4}$, $\mathrm{F}^{\epsilon}_{\epsilon,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_4,\beta_4}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\epsilon,\beta_1}$, $\mathrm{F}^{\epsilon}_{\beta_1,\epsilon,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_1,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_1,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_1,\epsilon,\beta_4}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_1,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_1,\epsilon}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_1,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_2,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_2,\beta_3}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_3,\epsilon}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_3,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_4,\epsilon}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\epsilon}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\epsilon}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\epsilon}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\epsilon}$, $\mathrm{F}^{\epsilon}_{\alpha_+,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\epsilon}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\epsilon}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\epsilon}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\epsilon}$, $\mathrm{F}^{\epsilon}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\epsilon}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\epsilon}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\epsilon}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\epsilon}$, $\mathrm{F}^{\epsilon}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\epsilon}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\epsilon}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\epsilon}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\epsilon}$, $\mathrm{F}^{\epsilon}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_2,\epsilon,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\beta_2,\epsilon,\beta_2}$, $\mathrm{F}^{\epsilon}_{\beta_2,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_2,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_2,\epsilon,\beta_4}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_1,\beta_3}$, $\mathrm{F}^{\epsilon}_{\beta_2,\alpha_+,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\epsilon}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_2,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_2,\beta_2}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_3,\beta_1}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_3,\epsilon,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_3,\epsilon,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\epsilon,\beta_3}$, $\mathrm{F}^{\epsilon}_{\beta_3,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_3,\epsilon,\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_1,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_1,\beta_2}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_1,\beta_4}$, $\mathrm{F}^{\epsilon}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\epsilon}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_2,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_3,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_4,\epsilon}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_4,\epsilon}$, $\mathrm{F}^{\beta_2}_{\beta_4,\epsilon,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_4,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_4,\epsilon,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\epsilon,\beta_4}$, $\mathrm{F}^{\epsilon}_{\beta_4,\epsilon,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_1,\epsilon}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_1,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_1,\beta_3}$, $\mathrm{F}^{\epsilon}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\epsilon}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_2,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_3,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_4,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_4,\beta_4}$

- $F^d_{abc} = \begin{pmatrix} 1 \end{pmatrix}$

  - $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\mathrm{I},\mathrm{I}}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\mathrm{I},\epsilon}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\mathrm{I},\beta_1}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\mathrm{I},\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\mathrm{I},\alpha_-}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\mathrm{I},\beta_2}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\mathrm{I},\beta_3}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\mathrm{I},\beta_4}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\epsilon,\mathrm{I}}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\epsilon,\epsilon}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\epsilon,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\epsilon,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\epsilon,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\epsilon,\beta_2}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\epsilon,\beta_3}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\epsilon,\beta_4}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_1,\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_1,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\beta_1,\beta_1}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\beta_1,\beta_1}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_1,\alpha_+}$,

$\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_1,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_1,\beta_2}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_1,\beta_2}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_1,\beta_3}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_1,\beta_3}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\epsilon}$,
$\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\alpha_+,\alpha_-}$,
$\mathrm{F}^{\beta_2}_{\mathrm{I},\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\beta_4}$,
$\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_-,\mathrm{I}}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_-,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_-,\beta_1}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\alpha_-,\alpha_+}$,
$\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_-,\beta_3}$,
$\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_2,\mathrm{I}}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_2,\epsilon}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_2,\beta_1}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_2,\alpha_-}$,
$\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\beta_2,\beta_2}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\beta_2,\beta_2}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_2,\beta_2}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_2,\beta_3}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_2,\beta_3}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_3,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_3,\epsilon}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_3,\beta_1}$,
$\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_3,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_3,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\beta_3,\beta_3}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\beta_3,\beta_3}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_3,\beta_3}$,
$\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_3,\beta_4}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_3,\beta_4}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_4,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_4,\epsilon}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_4,\beta_1}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_4,\alpha_-}$,
$\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_4,\beta_2}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_4,\beta_2}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_4,\beta_3}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_4,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\beta_4,\beta_4}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\beta_4,\beta_4}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_4,\beta_4}$, $\mathrm{F}^{\epsilon}_{\epsilon,\mathrm{I},\mathrm{I}}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\mathrm{I},\epsilon}$, $\mathrm{F}^{\beta_1}_{\epsilon,\mathrm{I},\beta_1}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\mathrm{I},\alpha_+}$,
$\mathrm{F}^{\alpha_+}_{\epsilon,\mathrm{I},\alpha_-}$, $\mathrm{F}^{\beta_2}_{\epsilon,\mathrm{I},\beta_2}$, $\mathrm{F}^{\beta_3}_{\epsilon,\mathrm{I},\beta_3}$, $\mathrm{F}^{\beta_4}_{\epsilon,\mathrm{I},\beta_4}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\epsilon,\mathrm{I}}$, $\mathrm{F}^{\epsilon}_{\epsilon,\epsilon,\epsilon}$, $\mathrm{F}^{\beta_1}_{\epsilon,\epsilon,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\epsilon,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\epsilon,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\epsilon,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_3}_{\epsilon,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_4}_{\epsilon,\epsilon,\beta_4}$,
$\mathrm{F}^{\beta_1}_{\epsilon,\beta_1,\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_1,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\beta_1,\beta_1}$, $\mathrm{F}^{\epsilon}_{\epsilon,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_1,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_1,\beta_2}$,
$\mathrm{F}^{\beta_2}_{\epsilon,\beta_1,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_+,\mathrm{I}}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_+,\beta_1}$, $\mathrm{F}^{\epsilon}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\beta_3}$,
$\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_-,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\epsilon,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_-,\beta_2}$,
$\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_2,\mathrm{I}}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_2,\epsilon}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_2,\alpha_+}$,
$\mathrm{F}^{\alpha_+}_{\epsilon,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_3,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_3,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_3,\alpha_-}$,
$\mathrm{F}^{\beta_1}_{\epsilon,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_4,\epsilon}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_4,\alpha_-}$,
$\mathrm{F}^{\beta_1}_{\epsilon,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_1,\mathrm{I},\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\beta_1,\mathrm{I},\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\mathrm{I},\beta_1}$, $\mathrm{F}^{\epsilon}_{\beta_1,\mathrm{I},\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_1,\mathrm{I},\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\mathrm{I},\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\mathrm{I},\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\mathrm{I},\alpha_-}$,
$\mathrm{F}^{\alpha_-}_{\beta_1,\mathrm{I},\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_1,\mathrm{I},\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_1,\mathrm{I},\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_1,\mathrm{I},\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_1,\mathrm{I},\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_1,\mathrm{I},\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_1,\mathrm{I},\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_1,\epsilon,\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\beta_1,\epsilon,\epsilon}$, $\mathrm{F}^{\beta_1}_{\beta_1,\epsilon,\beta_1}$,
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$\mathrm{F}^{\beta_2}_{\beta_3,\mathrm{I},\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_3,\epsilon,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\beta_3,\epsilon,\epsilon}$, $\mathrm{F}^{\beta_4}_{\beta_3,\epsilon,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\epsilon,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\epsilon,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\epsilon,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\epsilon,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_3,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_3,\epsilon,\beta_3}$,
$\mathrm{F}^{\beta_1}_{\beta_3,\epsilon,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_1,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_1,\mathrm{I}}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_1,\epsilon}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_1,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_1,\beta_2}$,
$\mathrm{F}^{\beta_3}_{\beta_3,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_1,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\mathrm{I}}$,
$\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\mathrm{I}}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\alpha_-,\alpha_+}$,
$\mathrm{F}^{\mathrm{I}}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_2,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_2,\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_2,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_2,\beta_2}$,
$\mathrm{F}^{\beta_4}_{\beta_3,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_2,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_2,\beta_4}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_2,\beta_4}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_3,\mathrm{I}}$,
$\mathrm{F}^{\epsilon}_{\beta_3,\beta_3,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_3,\mathrm{I}}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_3,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_3,\epsilon}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_3,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_3,\beta_3}$,
$\mathrm{F}^{\beta_1}_{\beta_3,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_4,\beta_1}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_4,\beta_1}$,
$\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_4,\beta_2}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_4,\beta_4}$,
$\mathrm{F}^{\beta_4}_{\beta_4,\mathrm{I},\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\beta_4,\mathrm{I},\epsilon}$, $\mathrm{F}^{\beta_2}_{\beta_4,\mathrm{I},\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_4,\mathrm{I},\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\mathrm{I},\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\mathrm{I},\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\mathrm{I},\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\mathrm{I},\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_4,\mathrm{I},\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_4,\mathrm{I},\beta_2}$,
$\mathrm{F}^{\beta_1}_{\beta_4,\mathrm{I},\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_4,\mathrm{I},\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\mathrm{I},\beta_4}$, $\mathrm{F}^{\epsilon}_{\beta_4,\mathrm{I},\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_4,\mathrm{I},\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_4,\epsilon,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\beta_4,\epsilon,\epsilon}$, $\mathrm{F}^{\beta_3}_{\beta_4,\epsilon,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\epsilon,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\epsilon,\alpha_+}$,
$\mathrm{F}^{\alpha_+}_{\beta_4,\epsilon,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\epsilon,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_4,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_4,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_4,\epsilon,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_1,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_1,\mathrm{I}}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_1,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_1,\beta_2}$,
$\mathrm{F}^{\epsilon}_{\beta_4,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_1,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_1,\beta_4}$,
$\mathrm{F}^{\beta_3}_{\beta_4,\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\mathrm{I}}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\mathrm{I}}$,
$\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_2,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_2,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_2,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_2,\beta_1}$,
$\mathrm{F}^{\beta_2}_{\beta_4,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_2,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_2,\beta_3}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_2,\beta_3}$,

$\mathrm{F}^{\beta_1}_{\beta_4,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_3,\mathrm{I}}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_3,\mathrm{I}}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_3,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_3,\beta_1}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_3,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_3,\beta_4}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_4,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_4,\epsilon}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_4,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_4,\beta_4}$

A.2.2 Pivotal Structures

There are 2 pivotal structures.

- $\epsilon_{\mathrm{I}} = 1$, $\epsilon_{\epsilon} = 1$, $\epsilon_{\beta_1} = 1$, $\epsilon_{\alpha_+} = 1$, $\epsilon_{\alpha_-} = 1$, $\epsilon_{\beta_2} = 1$, $\epsilon_{\beta_3} = 1$, and $\epsilon_{\beta_4} = 1$ which is spherical with quantum dimensions $q_{\mathrm{I}} = 1$, $q_{\epsilon} = 1$, $q_{\beta_1} = 2$, $q_{\alpha_+} = -3$, $q_{\alpha_-} = -3$, $q_{\beta_2} = 2$, $q_{\beta_3} = 2$, and $q_{\beta_4} = 2$.

- $\epsilon_{\mathrm{I}} = 1$, $\epsilon_{\epsilon} = 1$, $\epsilon_{\beta_1} = 1$, $\epsilon_{\alpha_+} = -1$, $\epsilon_{\alpha_-} = -1$, $\epsilon_{\beta_2} = 1$, $\epsilon_{\beta_3} = 1$, and $\epsilon_{\beta_4} = 1$ which is spherical with quantum dimensions $q_{\mathrm{I}} = 1$, $q_{\epsilon} = 1$, $q_{\beta_1} = 2$, $q_{\alpha_+} = 3$, $q_{\alpha_-} = 3$, $q_{\beta_2} = 2$, $q_{\beta_3} = 2$, and $q_{\beta_4} = 2$.

A.2.3 Braiding

Up to braided monoidal equivalence, there is one braidings which correspond to this solution.

- $\mathrm{R}^{\beta_2}_{\alpha_+,\alpha_-} = \mathrm{R}^{\beta_2}_{\alpha_-,\alpha_-} = \begin{pmatrix} (-1)^{5/6} \end{pmatrix}$
- $\mathrm{R}^{\beta_1}_{\alpha_+,\alpha_-} = \mathrm{R}^{\beta_1}_{\alpha_-,\alpha_-} = \begin{pmatrix} \sqrt[6]{-1} \end{pmatrix}$
- $R^{c}_{a,b} = \begin{pmatrix} -i \end{pmatrix}$
  - $\mathrm{R}^{\epsilon}_{\alpha_+,\alpha_-}$, $\mathrm{R}^{\beta_3}_{\alpha_+,\alpha_-}$, $\mathrm{R}^{\beta_4}_{\alpha_+,\alpha_-}$, $\mathrm{R}^{\mathrm{I}}_{\alpha_-,\alpha_-}$, $\mathrm{R}^{\beta_3}_{\alpha_-,\alpha_-}$, $\mathrm{R}^{\beta_4}_{\alpha_-,\alpha_-}$
- $\mathrm{R}^{\beta_2}_{\alpha_+,\alpha_+} = \mathrm{R}^{\beta_2}_{\alpha_-,\alpha_+} = \begin{pmatrix} -(-1)^{5/6} \end{pmatrix}$
- $\mathrm{R}^{\beta_1}_{\alpha_+,\alpha_+} = \mathrm{R}^{\beta_1}_{\alpha_-,\alpha_+} = \begin{pmatrix} -\sqrt[6]{-1} \end{pmatrix}$

- $R^{c}_{a,b} = \begin{pmatrix} i \end{pmatrix}$
  - $\mathrm{R}^{\mathrm{I}}_{\alpha_+,\alpha_+}$, $\mathrm{R}^{\beta_3}_{\alpha_+,\alpha_+}$, $\mathrm{R}^{\beta_4}_{\alpha_+,\alpha_+}$, $\mathrm{R}^{\epsilon}_{\alpha_-,\alpha_+}$, $\mathrm{R}^{\beta_3}_{\alpha_-,\alpha_+}$, $\mathrm{R}^{\beta_4}_{\alpha_-,\alpha_+}$
- $\mathrm{R}^{\alpha_-}_{\alpha_+,\beta_1} = \mathrm{R}^{\alpha_+}_{\alpha_-,\beta_1} = \mathrm{R}^{\epsilon}_{\beta_2,\beta_2} = \begin{pmatrix} \frac{1}{2}\left(1 + i\sqrt{3}\right) \end{pmatrix}$
- $R^{c}_{a,b} = \begin{pmatrix} -1 \end{pmatrix}$
  - $\mathrm{R}^{\alpha_-}_{\alpha_+,\epsilon}$, $\mathrm{R}^{\alpha_-}_{\alpha_+,\beta_3}$, $\mathrm{R}^{\alpha_-}_{\alpha_+,\beta_4}$, $\mathrm{R}^{\alpha_+}_{\alpha_-,\epsilon}$, $\mathrm{R}^{\alpha_+}_{\alpha_-,\beta_3}$, $\mathrm{R}^{\alpha_+}_{\alpha_-,\beta_4}$, $\mathrm{R}^{\epsilon}_{\beta_3,\beta_3}$, $\mathrm{R}^{\epsilon}_{\beta_4,\beta_4}$
- $R^{c}_{a,b} = \begin{pmatrix} -\frac{1}{2}i\left(-i + \sqrt{3}\right) \end{pmatrix}$
  - $\mathrm{R}^{\beta_1}_{\beta_1,\beta_1}$, $\mathrm{R}^{\alpha_+}_{\beta_1,\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\beta_1,\alpha_+}$, $\mathrm{R}^{\alpha_+}_{\beta_1,\alpha_-}$, $\mathrm{R}^{\alpha_-}_{\beta_1,\alpha_-}$, $\mathrm{R}^{\beta_4}_{\beta_1,\beta_3}$, $\mathrm{R}^{\beta_3}_{\beta_1,\beta_4}$, $\mathrm{R}^{\alpha_+}_{\alpha_+,\beta_1}$, $\mathrm{R}^{\alpha_-}_{\alpha_-,\beta_1}$, $\mathrm{R}^{\mathrm{I}}_{\beta_2,\beta_2}$, $\mathrm{R}^{\beta_1}_{\beta_2,\beta_3}$, $\mathrm{R}^{\beta_1}_{\beta_2,\beta_4}$, $\mathrm{R}^{\beta_4}_{\beta_3,\beta_1}$, $\mathrm{R}^{\beta_1}_{\beta_3,\beta_2}$, $\mathrm{R}^{\beta_2}_{\beta_3,\beta_4}$, $\mathrm{R}^{\beta_3}_{\beta_4,\beta_1}$, $\mathrm{R}^{\beta_1}_{\beta_4,\beta_2}$, $\mathrm{R}^{\beta_2}_{\beta_4,\beta_3}$
- $\mathrm{R}^{\epsilon}_{\beta_1,\beta_1} = \mathrm{R}^{\alpha_-}_{\alpha_+,\beta_2} = \mathrm{R}^{\alpha_+}_{\alpha_-,\beta_2} = \begin{pmatrix} \frac{1}{2}\left(1 - i\sqrt{3}\right) \end{pmatrix}$
- $R^{c}_{a,b} = \begin{pmatrix} \frac{1}{2}i\left(i + \sqrt{3}\right) \end{pmatrix}$
  - $\mathrm{R}^{\mathrm{I}}_{\beta_1,\beta_1}$, $\mathrm{R}^{\beta_2}_{\beta_1,\beta_3}$, $\mathrm{R}^{\beta_2}_{\beta_1,\beta_4}$, $\mathrm{R}^{\alpha_+}_{\alpha_+,\beta_2}$, $\mathrm{R}^{\alpha_-}_{\alpha_-,\beta_2}$, $\mathrm{R}^{\alpha_+}_{\beta_2,\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\beta_2,\alpha_+}$, $\mathrm{R}^{\alpha_+}_{\beta_2,\alpha_-}$, $\mathrm{R}^{\alpha_-}_{\beta_2,\alpha_-}$, $\mathrm{R}^{\beta_2}_{\beta_2,\beta_2}$, $\mathrm{R}^{\beta_4}_{\beta_2,\beta_3}$, $\mathrm{R}^{\beta_3}_{\beta_2,\beta_4}$, $\mathrm{R}^{\beta_2}_{\beta_3,\beta_1}$, $\mathrm{R}^{\beta_4}_{\beta_3,\beta_2}$, $\mathrm{R}^{\beta_1}_{\beta_3,\beta_4}$, $\mathrm{R}^{\beta_2}_{\beta_4,\beta_1}$, $\mathrm{R}^{\beta_3}_{\beta_4,\beta_2}$, $\mathrm{R}^{\beta_1}_{\beta_4,\beta_3}$
- $R^{c}_{a,b} = \begin{pmatrix} 1 \end{pmatrix}$
  - $\mathrm{R}^{\mathrm{I}}_{\mathrm{I},\mathrm{I}}$, $\mathrm{R}^{\epsilon}_{\mathrm{I},\epsilon}$, $\mathrm{R}^{\beta_1}_{\mathrm{I},\beta_1}$, $\mathrm{R}^{\alpha_+}_{\mathrm{I},\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\mathrm{I},\alpha_-}$, $\mathrm{R}^{\beta_2}_{\mathrm{I},\beta_2}$, $\mathrm{R}^{\beta_3}_{\mathrm{I},\beta_3}$, $\mathrm{R}^{\beta_4}_{\mathrm{I},\beta_4}$, $\mathrm{R}^{\epsilon}_{\epsilon,\mathrm{I}}$, $\mathrm{R}^{\mathrm{I}}_{\epsilon,\epsilon}$, $\mathrm{R}^{\beta_1}_{\epsilon,\beta_1}$, $\mathrm{R}^{\alpha_-}_{\epsilon,\alpha_+}$, $\mathrm{R}^{\alpha_+}_{\epsilon,\alpha_-}$, $\mathrm{R}^{\beta_2}_{\epsilon,\beta_2}$, $\mathrm{R}^{\beta_3}_{\epsilon,\beta_3}$, $\mathrm{R}^{\beta_4}_{\epsilon,\beta_4}$, $\mathrm{R}^{\beta_1}_{\beta_1,\mathrm{I}}$, $\mathrm{R}^{\beta_1}_{\beta_1,\epsilon}$, $\mathrm{R}^{\beta_3}_{\beta_1,\beta_2}$, $\mathrm{R}^{\beta_4}_{\beta_1,\beta_2}$, $\mathrm{R}^{\alpha_+}_{\alpha_+,\mathrm{I}}$, $\mathrm{R}^{\alpha_+}_{\alpha_+,\beta_3}$, $\mathrm{R}^{\alpha_+}_{\alpha_+,\beta_4}$, $\mathrm{R}^{\alpha_-}_{\alpha_-,\mathrm{I}}$, $\mathrm{R}^{\alpha_-}_{\alpha_-,\beta_3}$, $\mathrm{R}^{\alpha_-}_{\alpha_-,\beta_4}$, $\mathrm{R}^{\beta_2}_{\beta_2,\mathrm{I}}$, $\mathrm{R}^{\beta_2}_{\beta_2,\epsilon}$, $\mathrm{R}^{\beta_3}_{\beta_2,\beta_1}$, $\mathrm{R}^{\beta_4}_{\beta_2,\beta_1}$, $\mathrm{R}^{\beta_3}_{\beta_3,\mathrm{I}}$, $\mathrm{R}^{\beta_3}_{\beta_3,\epsilon}$, $\mathrm{R}^{\alpha_+}_{\beta_3,\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\beta_3,\alpha_+}$, $\mathrm{R}^{\alpha_+}_{\beta_3,\alpha_-}$, $\mathrm{R}^{\alpha_-}_{\beta_3,\alpha_-}$, $\mathrm{R}^{\mathrm{I}}_{\beta_3,\beta_3}$, $\mathrm{R}^{\beta_3}_{\beta_3,\beta_3}$, $\mathrm{R}^{\beta_4}_{\beta_4,\mathrm{I}}$, $\mathrm{R}^{\beta_4}_{\beta_4,\epsilon}$, $\mathrm{R}^{\alpha_+}_{\beta_4,\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\beta_4,\alpha_+}$, $\mathrm{R}^{\alpha_+}_{\beta_4,\alpha_-}$, $\mathrm{R}^{\alpha_-}_{\beta_4,\alpha_-}$, $\mathrm{R}^{\mathrm{I}}_{\beta_4,\beta_4}$, $\mathrm{R}^{\beta_4}_{\beta_4,\beta_4}$

A.2.4 Balanced, Ribbon, and Modular Structures

There are 2 balanced categories which correspond to this solution. Both of them are modular.

Balanced Category 1 has pivotal structure 1 and braiding 1. Its T-Matrix is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) \end{pmatrix}.$$

It is modular and has S-Matrix given by:

$$\begin{pmatrix} 1 & 1 & 2 & -3 & -3 & 2 & 2 & 2 \\ 1 & 1 & 2 & 3 & 3 & 2 & 2 & 2 \\ 2 & 2 & 4 & 0 & 0 & -2 & -2 & -2 \\ -3 & 3 & 0 & -3 & 3 & 0 & 0 & 0 \\ -3 & 3 & 0 & 3 & -3 & 0 & 0 & 0 \\ 2 & 2 & -2 & 0 & 0 & 4 & -2 & -2 \\ 2 & 2 & -2 & 0 & 0 & -2 & -2 & 4 \\ 2 & 2 & -2 & 0 & 0 & -2 & 4 & -2 \end{pmatrix}.$$

Balanced Category 2 has pivotal structure 2. Its T-Matrix is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) \end{pmatrix}.$$

It is modular and has S-Matrix given by:

$$\begin{pmatrix} 1 & 1 & 2 & 3 & 3 & 2 & 2 & 2 \\ 1 & 1 & 2 & -3 & -3 & 2 & 2 & 2 \\ 2 & 2 & 4 & 0 & 0 & -2 & -2 & -2 \\ 3 & -3 & 0 & -3 & 3 & 0 & 0 & 0 \\ 3 & -3 & 0 & 3 & -3 & 0 & 0 & 0 \\ 2 & 2 & -2 & 0 & 0 & 4 & -2 & -2 \\ 2 & 2 & -2 & 0 & 0 & -2 & -2 & 4 \\ 2 & 2 & -2 & 0 & 0 & -2 & 4 & -2 \end{pmatrix}.$$

A.3 Solution 3 – Non-Group Theoretical Category 1

A.3.1 F-Matrices

- $\mathrm{F}^{\alpha_+}_{\alpha_+,\alpha_-,\alpha_-} = \mathrm{F}^{\alpha_+}_{\alpha_-,\alpha_-,\alpha_+} = \mathrm{F}^{\alpha_-}_{\alpha_-,\alpha_-,\alpha_-} = \begin{pmatrix} \frac{1}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{\sqrt{2}}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$

- $\mathrm{F}^{\alpha_-}_{\alpha_+,\alpha_-,\alpha_+} = \begin{pmatrix} -\frac{1}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{\sqrt{2}}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$

- $\mathrm{F}^{\alpha_-}_{\alpha_+,\alpha_+,\alpha_-} = \mathrm{F}^{\alpha_-}_{\alpha_-,\alpha_+,\alpha_+} = \mathrm{F}^{\alpha_+}_{\alpha_-,\alpha_+,\alpha_-} = \begin{pmatrix} -\frac{1}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$

- $\mathrm{F}^{\alpha_+}_{\alpha_+,\alpha_+,\alpha_+} = \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{\sqrt{2}}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{\sqrt{2}}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$

- $\mathrm{F}^{\alpha_-}_{\alpha_-,\alpha_+,\alpha_-} = \mathrm{F}^{\alpha_+}_{\alpha_-,\alpha_-,\alpha_-} = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix}$
- $F^d_{abc} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_-}_{\alpha_+,\alpha_+,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\alpha_+,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\alpha_-,\alpha_+}$
- $\mathrm{F}^{\beta_1}_{\beta_1,\beta_1,\beta_1} = \mathrm{F}^{\beta_2}_{\beta_2,\beta_2,\beta_2} = \mathrm{F}^{\beta_3}_{\beta_3,\beta_3,\beta_3} = \mathrm{F}^{\beta_4}_{\beta_4,\beta_4,\beta_4} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$
- $F^d_{abc} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$
  - $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_4,\alpha_+}$
- $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_1,\alpha_+} = \mathrm{F}^{\beta_1}_{\alpha_-,\beta_2,\alpha_+} = \mathrm{F}^{\beta_4}_{\alpha_-,\beta_3,\alpha_+} = \mathrm{F}^{\beta_3}_{\alpha_-,\beta_4,\alpha_+} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_1,\alpha_+} = \mathrm{F}^{\beta_4}_{\alpha_-,\beta_1,\alpha_+} = \mathrm{F}^{\beta_2}_{\alpha_-,\beta_2,\alpha_+} = \mathrm{F}^{\beta_1}_{\alpha_-,\beta_4,\alpha_+} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$
- $F^d_{abc} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$
  - $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_4,\alpha_-}$

- $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_1,\alpha_-} = \mathrm{F}^{\beta_1}_{\alpha_+,\beta_2,\alpha_-} = \mathrm{F}^{\beta_4}_{\alpha_+,\beta_3,\alpha_-} = \mathrm{F}^{\beta_3}_{\alpha_+,\beta_4,\alpha_-} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
- $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_1,\alpha_-} = \mathrm{F}^{\beta_4}_{\alpha_+,\beta_1,\alpha_-} = \mathrm{F}^{\beta_2}_{\alpha_+,\beta_2,\alpha_-} = \mathrm{F}^{\beta_1}_{\alpha_+,\beta_4,\alpha_-} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$
- $F^d_{abc} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\beta_4}$
- $F^d_{abc} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
  - $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\beta_3}$
- $F^d_{abc} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\beta_1}$
- $F^d_{abc} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\beta_4}$
- $F^d_{abc} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

- $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\beta_4}$

- $F^d_{abc} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

  - $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_1,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_2,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_2,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_1,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_3,\beta_4}$

- $F^d_{abc} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

  - $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\beta_3}$

- $F^d_{abc} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

  - $\mathrm{F}^{\beta_2}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_4,\alpha_-,\alpha_-}$

- $F^d_{abc} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

  - $\mathrm{F}^{\beta_1}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_2,\alpha_-}$

- $F^d_{abc} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\beta_1}$
- $F^d_{abc} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\beta_1}$
- $F^d_{abc} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$
  - $\mathrm{F}^{\beta_4}_{\beta_1,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_4,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\beta_1}$,
    $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\beta_4}$,
    $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_1,\alpha_+}$,
    $\mathrm{F}^{\beta_1}_{\beta_2,\alpha_+,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_2,\alpha_-}$,
    $\mathrm{F}^{\beta_4}_{\beta_3,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_2,\alpha_-}$,
    $\mathrm{F}^{\beta_4}_{\beta_4,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_4,\beta_3}$
- $F^d_{abc} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$
  - $\mathrm{F}^{\beta_2}_{\beta_1,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\beta_1}$,
    $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\beta_1}$,
    $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_1,\alpha_-}$,
    $\mathrm{F}^{\beta_4}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_2,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_2,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_1,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_2,\alpha_+}$,
    $\mathrm{F}^{\beta_1}_{\beta_3,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_4,\beta_2}$
- $F^d_{abc} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

– $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_4,\alpha_+}$

- $F^d_{abc} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

  – $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_2,\alpha_+,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_4,\alpha_+}$

- $F^d_{abc} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

  – $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_4,\alpha_-}$

- $F^d_{abc} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

  – $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_2,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_2,\alpha_+,\alpha_+}$,

$\mathrm{F}^{\beta_4}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_2}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_4,\alpha_-}$

- $F^d_{abc} = \begin{pmatrix} -1 \end{pmatrix}$

  - $\mathrm{F}^{\beta_1}_{\epsilon,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_1,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_1,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\beta_1}$, $\mathrm{F}^{\epsilon}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\epsilon,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\epsilon,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\epsilon,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\epsilon,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\epsilon,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\epsilon,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\epsilon,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\epsilon,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_3,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_1,\epsilon,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_1,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_1,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_3}_{\beta_1,\epsilon,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_1,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_2,\epsilon}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_2,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_3,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_3,\beta_2}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_4,\epsilon}$, $\mathrm{F}^{\epsilon}_{\alpha_+,\epsilon,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\alpha_+,\epsilon,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\epsilon}$, $\mathrm{F}^{\epsilon}_{\alpha_+,\alpha_+,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_-,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\alpha_-,\epsilon,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\alpha_-,\epsilon,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\epsilon}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\epsilon}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\epsilon}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\epsilon}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\epsilon}$, $\mathrm{F}^{\epsilon}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\epsilon}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\epsilon}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\epsilon}_{\alpha_-,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\epsilon}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\epsilon}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\epsilon}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\epsilon}$, $\mathrm{F}^{\beta_3}_{\beta_2,\epsilon,\beta_1}$, $\mathrm{F}^{\beta_2}_{\beta_2,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_2,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_2,\epsilon,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_2,\beta_1,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_1,\beta_3}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_2,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_3,\epsilon}$, $\mathrm{F}^{\beta_4}_{\beta_2,\beta_3,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_2,\beta_4,\epsilon}$, $\mathrm{F}^{\beta_2}_{\beta_3,\epsilon,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_3,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_3,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_3,\epsilon,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_1,\epsilon}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_1,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_1,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_2,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_2,\beta_1}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_3,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_4,\epsilon}$, $\mathrm{F}^{\beta_3}_{\beta_4,\epsilon,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_4,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_4,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_4,\epsilon,\beta_4}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_1,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_2,\beta_3}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_4,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_4,\beta_4}$

- $F^d_{abc} = \begin{pmatrix} 1 \end{pmatrix}$

  - $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\mathrm{I},\mathrm{I}}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\mathrm{I},\epsilon}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\mathrm{I},\beta_1}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\mathrm{I},\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\mathrm{I},\alpha_-}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\mathrm{I},\beta_2}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\mathrm{I},\beta_3}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\mathrm{I},\beta_4}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\epsilon,\mathrm{I}}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\epsilon,\epsilon}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\epsilon,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\epsilon,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\epsilon,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\epsilon,\beta_2}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\epsilon,\beta_3}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\epsilon,\beta_4}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_1,\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_1,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\beta_1,\beta_1}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\beta_1,\beta_1}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_1,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_1,\beta_2}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_1,\beta_2}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_1,\beta_3}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_1,\beta_3}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\epsilon}$,

$\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\alpha_+,\alpha_-}$,
$\mathrm{F}^{\beta_2}_{\mathrm{I},\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\beta_4}$,
$\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_-,\mathrm{I}}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_-,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_-,\beta_1}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\alpha_-,\alpha_+}$,
$\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_-,\beta_3}$,
$\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_2,\mathrm{I}}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_2,\epsilon}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_2,\beta_1}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_2,\alpha_-}$,
$\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\beta_2,\beta_2}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\beta_2,\beta_2}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_2,\beta_2}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_2,\beta_3}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_2,\beta_3}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_3,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_3,\epsilon}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_3,\beta_1}$,
$\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_3,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_3,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\beta_3,\beta_3}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\beta_3,\beta_3}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_3,\beta_3}$,
$\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_3,\beta_4}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_3,\beta_4}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_4,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_4,\epsilon}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_4,\beta_1}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_4,\alpha_-}$,
$\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_4,\beta_2}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_4,\beta_2}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_4,\beta_3}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_4,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\beta_4,\beta_4}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\beta_4,\beta_4}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_4,\beta_4}$, $\mathrm{F}^{\epsilon}_{\epsilon,\mathrm{I},\mathrm{I}}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\mathrm{I},\epsilon}$, $\mathrm{F}^{\beta_1}_{\epsilon,\mathrm{I},\beta_1}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\mathrm{I},\alpha_+}$,
$\mathrm{F}^{\alpha_+}_{\epsilon,\mathrm{I},\alpha_-}$, $\mathrm{F}^{\beta_2}_{\epsilon,\mathrm{I},\beta_2}$, $\mathrm{F}^{\beta_3}_{\epsilon,\mathrm{I},\beta_3}$, $\mathrm{F}^{\beta_4}_{\epsilon,\mathrm{I},\beta_4}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\epsilon,\mathrm{I}}$, $\mathrm{F}^{\epsilon}_{\epsilon,\epsilon,\epsilon}$, $\mathrm{F}^{\beta_1}_{\epsilon,\epsilon,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\epsilon,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\epsilon,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\epsilon,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_3}_{\epsilon,\epsilon,\beta_3}$,
$\mathrm{F}^{\beta_4}_{\epsilon,\epsilon,\beta_4}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_1,\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_1,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\beta_1,\beta_1}$, $\mathrm{F}^{\epsilon}_{\epsilon,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_+,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_+,\beta_1}$,
$\mathrm{F}^{\beta_1}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_-,\mathrm{I}}$,
$\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_1}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\epsilon,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_-,\beta_4}$,
$\mathrm{F}^{\beta_2}_{\epsilon,\beta_2,\mathrm{I}}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_2,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\beta_2,\beta_2}$, $\mathrm{F}^{\epsilon}_{\epsilon,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_3,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_3,\epsilon}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_3,\beta_1}$,
$\mathrm{F}^{\mathrm{I}}_{\epsilon,\beta_3,\beta_3}$, $\mathrm{F}^{\epsilon}_{\epsilon,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_4,\epsilon}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_4,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\beta_4,\beta_4}$,
$\mathrm{F}^{\epsilon}_{\epsilon,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_1,\mathrm{I},\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\beta_1,\mathrm{I},\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\mathrm{I},\beta_1}$, $\mathrm{F}^{\epsilon}_{\beta_1,\mathrm{I},\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_1,\mathrm{I},\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\mathrm{I},\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\mathrm{I},\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\mathrm{I},\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\mathrm{I},\alpha_-}$,
$\mathrm{F}^{\beta_3}_{\beta_1,\mathrm{I},\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_1,\mathrm{I},\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_1,\mathrm{I},\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_1,\mathrm{I},\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_1,\mathrm{I},\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_1,\mathrm{I},\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_1,\epsilon,\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\beta_1,\epsilon,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\epsilon,\beta_1}$, $\mathrm{F}^{\epsilon}_{\beta_1,\epsilon,\beta_1}$,
$\mathrm{F}^{\alpha_+}_{\beta_1,\epsilon,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\epsilon,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\epsilon,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\epsilon,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_1,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_1,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_1,\epsilon,\beta_4}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_1,\mathrm{I}}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_1,\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_1,\mathrm{I}}$,
$\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_1,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_1,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_1,\beta_4}$,
$\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\mathrm{I}}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\epsilon}$,
$\mathrm{F}^{\mathrm{I}}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_2,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_2,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_2,\beta_2}$,
$\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_2,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_2,\beta_4}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_2,\beta_4}$,
$\mathrm{F}^{\beta_2}_{\beta_1,\beta_3,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_3,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_3,\epsilon}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_3,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_3,\beta_2}$,
$\mathrm{F}^{\beta_2}_{\beta_1,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_3,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_4,\epsilon}$,
$\mathrm{F}^{\beta_2}_{\beta_1,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_4,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_4,\beta_2}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_4,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_4,\beta_3}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_4,\beta_3}$,
$\mathrm{F}^{\beta_1}_{\beta_1,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_4,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\mathrm{I},\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\mathrm{I},\epsilon}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\mathrm{I},\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\mathrm{I},\beta_1}$,

$$\mathrm{F}^{\mathrm{I}}_{\alpha_+,\mathrm{I},\alpha_+},\ \mathrm{F}^{\beta_1}_{\alpha_+,\mathrm{I},\alpha_+},\ \mathrm{F}^{\beta_2}_{\alpha_+,\mathrm{I},\alpha_+},\ \mathrm{F}^{\beta_3}_{\alpha_+,\mathrm{I},\alpha_+},\ \mathrm{F}^{\beta_4}_{\alpha_+,\mathrm{I},\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_+,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_1}_{\alpha_+,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_2}_{\alpha_+,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_3}_{\alpha_+,\mathrm{I},\alpha_-},$$

$$\mathrm{F}^{\beta_4}_{\alpha_+,\mathrm{I},\alpha_-},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\mathrm{I},\beta_2},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\mathrm{I},\beta_2},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\mathrm{I},\beta_3},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\mathrm{I},\beta_3},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\mathrm{I},\beta_4},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\mathrm{I},\beta_4},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\epsilon,\mathrm{I}},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\epsilon,\epsilon},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\epsilon,\beta_1},$$

$$\mathrm{F}^{\alpha_-}_{\alpha_+,\epsilon,\beta_1},\ \mathrm{F}^{\beta_1}_{\alpha_+,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_2}_{\alpha_+,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_3}_{\alpha_+,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_4}_{\alpha_+,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_1}_{\alpha_+,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_2}_{\alpha_+,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_3}_{\alpha_+,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_4}_{\alpha_+,\epsilon,\alpha_-},$$

$$\mathrm{F}^{\alpha_+}_{\alpha_+,\epsilon,\beta_2},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\epsilon,\beta_2},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\epsilon,\beta_3},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\epsilon,\beta_3},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\epsilon,\beta_4},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\epsilon,\beta_4},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_1,\alpha_+},$$

$$\mathrm{F}^{\epsilon}_{\alpha_+,\beta_1,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_1,\alpha_-},\ \mathrm{F}^{\epsilon}_{\alpha_+,\beta_1,\alpha_-},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\epsilon},$$

$$\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\epsilon},\ \mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\epsilon},\ \mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_+,\beta_1},\ \mathrm{F}^{\epsilon}_{\alpha_+,\alpha_+,\beta_1},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_+,\beta_2},\ \mathrm{F}^{\epsilon}_{\alpha_+,\alpha_+,\beta_2},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_+,\beta_3},$$

$$\mathrm{F}^{\epsilon}_{\alpha_+,\alpha_+,\beta_3},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_+,\beta_4},\ \mathrm{F}^{\epsilon}_{\alpha_+,\alpha_+,\beta_4},\ \mathrm{F}^{\epsilon}_{\alpha_+,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\epsilon},$$

$$\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\epsilon},\ \mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\epsilon},\ \mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_-,\beta_1},\ \mathrm{F}^{\epsilon}_{\alpha_+,\alpha_-,\beta_1},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_-,\beta_2},\ \mathrm{F}^{\epsilon}_{\alpha_+,\alpha_-,\beta_2},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_-,\beta_3},$$

$$\mathrm{F}^{\epsilon}_{\alpha_+,\alpha_-,\beta_3},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_-,\beta_4},\ \mathrm{F}^{\epsilon}_{\alpha_+,\alpha_-,\beta_4},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_2,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_+,\beta_2,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_2,\alpha_-},$$

$$\mathrm{F}^{\epsilon}_{\alpha_+,\beta_2,\alpha_-},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_3,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_+,\beta_3,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_3,\alpha_-},\ \mathrm{F}^{\epsilon}_{\alpha_+,\beta_3,\alpha_-},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\mathrm{I}},$$

$$\mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_4,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_+,\beta_4,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_4,\alpha_-},\ \mathrm{F}^{\epsilon}_{\alpha_+,\beta_4,\alpha_-},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\mathrm{I},\mathrm{I}},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\mathrm{I},\epsilon},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\mathrm{I},\beta_1},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\mathrm{I},\beta_1},\ \mathrm{F}^{\epsilon}_{\alpha_-,\mathrm{I},\alpha_+},$$

$$\mathrm{F}^{\beta_1}_{\alpha_-,\mathrm{I},\alpha_+},\ \mathrm{F}^{\beta_2}_{\alpha_-,\mathrm{I},\alpha_+},\ \mathrm{F}^{\beta_3}_{\alpha_-,\mathrm{I},\alpha_+},\ \mathrm{F}^{\beta_4}_{\alpha_-,\mathrm{I},\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_1}_{\alpha_-,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_2}_{\alpha_-,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_3}_{\alpha_-,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_4}_{\alpha_-,\mathrm{I},\alpha_-},$$

$$\mathrm{F}^{\alpha_+}_{\alpha_-,\mathrm{I},\beta_2},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\mathrm{I},\beta_2},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\mathrm{I},\beta_3},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\mathrm{I},\beta_3},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\mathrm{I},\beta_4},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\mathrm{I},\beta_4},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\epsilon,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\epsilon,\epsilon},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\epsilon,\beta_1},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\epsilon,\beta_1},$$

$$\mathrm{F}^{\beta_1}_{\alpha_-,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_2}_{\alpha_-,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_3}_{\alpha_-,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_4}_{\alpha_-,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_1}_{\alpha_-,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_2}_{\alpha_-,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_3}_{\alpha_-,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_4}_{\alpha_-,\epsilon,\alpha_-},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\epsilon,\beta_2},$$

$$\mathrm{F}^{\alpha_-}_{\alpha_-,\epsilon,\beta_2},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\epsilon,\beta_3},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\epsilon,\beta_3},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\epsilon,\beta_4},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\epsilon,\beta_4},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_1,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_1,\alpha_+},$$

$$\mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_1,\alpha_-},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_1,\alpha_-},\ \mathrm{F}^{\epsilon}_{\alpha_-,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_+,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_+,\beta_1},$$

$$\mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_+,\beta_2},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_+,\beta_3},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_+,\beta_4},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\epsilon}_{\alpha_-,\alpha_-,\epsilon},$$

$$\mathrm{F}^{\epsilon}_{\alpha_-,\alpha_-,\beta_1},\ \mathrm{F}^{\epsilon}_{\alpha_-,\alpha_-,\beta_2},\ \mathrm{F}^{\epsilon}_{\alpha_-,\alpha_-,\beta_3},\ \mathrm{F}^{\epsilon}_{\alpha_-,\alpha_-,\beta_4},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_2,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_2,\alpha_+},$$

$$\mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_2,\alpha_-},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_2,\alpha_-},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_3,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_3,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_3,\alpha_-},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_3,\alpha_-},$$

$$\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_4,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_4,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_4,\alpha_-},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_4,\alpha_-},\ \mathrm{F}^{\beta_2}_{\beta_2,\mathrm{I},\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\beta_2,\mathrm{I},\epsilon},\ \mathrm{F}^{\beta_3}_{\beta_2,\mathrm{I},\beta_1},$$

$$\mathrm{F}^{\beta_4}_{\beta_2,\mathrm{I},\beta_1},\ \mathrm{F}^{\alpha_+}_{\beta_2,\mathrm{I},\alpha_+},\ \mathrm{F}^{\alpha_-}_{\beta_2,\mathrm{I},\alpha_+},\ \mathrm{F}^{\alpha_+}_{\beta_2,\mathrm{I},\alpha_-},\ \mathrm{F}^{\alpha_-}_{\beta_2,\mathrm{I},\alpha_-},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\mathrm{I},\beta_2},\ \mathrm{F}^{\epsilon}_{\beta_2,\mathrm{I},\beta_2},\ \mathrm{F}^{\beta_2}_{\beta_2,\mathrm{I},\beta_2},\ \mathrm{F}^{\beta_1}_{\beta_2,\mathrm{I},\beta_3},\ \mathrm{F}^{\beta_4}_{\beta_2,\mathrm{I},\beta_3},$$

$$\mathrm{F}^{\beta_1}_{\beta_2,\mathrm{I},\beta_4},\ \mathrm{F}^{\beta_3}_{\beta_2,\mathrm{I},\beta_4},\ \mathrm{F}^{\beta_2}_{\beta_2,\epsilon,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\beta_2,\epsilon,\epsilon},\ \mathrm{F}^{\beta_4}_{\beta_2,\epsilon,\beta_1},\ \mathrm{F}^{\alpha_+}_{\beta_2,\epsilon,\alpha_+},\ \mathrm{F}^{\alpha_-}_{\beta_2,\epsilon,\alpha_+},\ \mathrm{F}^{\alpha_+}_{\beta_2,\epsilon,\alpha_-},\ \mathrm{F}^{\alpha_-}_{\beta_2,\epsilon,\alpha_-},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\epsilon,\beta_2},$$

$$\mathrm{F}^{\epsilon}_{\beta_2,\epsilon,\beta_2},\ \mathrm{F}^{\beta_4}_{\beta_2,\epsilon,\beta_3},\ \mathrm{F}^{\beta_3}_{\beta_2,\epsilon,\beta_4},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_1,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_1,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_1,\epsilon},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_1,\beta_1},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_1,\beta_1},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_1,\beta_2},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_1,\beta_2},$$

$$\mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_1,\beta_3},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_1,\beta_3},\ \mathrm{F}^{\beta_2}_{\beta_2,\beta_1,\beta_3},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_1,\beta_3},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_1,\beta_4},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_1,\beta_4},\ \mathrm{F}^{\beta_2}_{\beta_2,\beta_1,\beta_4},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_1,\beta_4},\ \mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\mathrm{I}},$$

$$\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\alpha_+,\alpha_+},\ \mathrm{F}^{\epsilon}_{\beta_2,\alpha_+,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\alpha_+,\alpha_-},\ \mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\alpha_-,\alpha_+},$$

$$\mathrm{F}^{\epsilon}_{\beta_2,\alpha_-,\alpha_+},\ \mathrm{F}^{\epsilon}_{\beta_2,\alpha_-,\alpha_-},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_2,\mathrm{I}},\ \mathrm{F}^{\epsilon}_{\beta_2,\beta_2,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\beta_2,\beta_2,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_2,\epsilon},\ \mathrm{F}^{\epsilon}_{\beta_2,\beta_2,\epsilon},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_2,\beta_1},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_2,\beta_1},$$

$\mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_2,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_2,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_3,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\beta_2,\beta_3,\mathrm{I}}$, $\mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_3,\beta_1}$,
$\mathrm{F}^{\beta_2}_{\beta_2,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_2,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_2,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_2,\beta_3,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_3,\beta_4}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_3,\beta_4}$,
$\mathrm{F}^{\beta_3}_{\beta_2,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_2,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\beta_2,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_4,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_4,\beta_1}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_4,\beta_1}$,
$\mathrm{F}^{\beta_4}_{\beta_2,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_2,\beta_4,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_4,\beta_3}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_3}_{\beta_2,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_2,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_4,\beta_4}$,
$\mathrm{F}^{\beta_3}_{\beta_2,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_3,\mathrm{I},\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\beta_3,\mathrm{I},\epsilon}$, $\mathrm{F}^{\beta_2}_{\beta_3,\mathrm{I},\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_3,\mathrm{I},\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\mathrm{I},\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\mathrm{I},\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\mathrm{I},\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\mathrm{I},\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_3,\mathrm{I},\beta_2}$,
$\mathrm{F}^{\beta_4}_{\beta_3,\mathrm{I},\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\mathrm{I},\beta_3}$, $\mathrm{F}^{\epsilon}_{\beta_3,\mathrm{I},\beta_3}$, $\mathrm{F}^{\beta_3}_{\beta_3,\mathrm{I},\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_3,\mathrm{I},\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_3,\mathrm{I},\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_3,\epsilon,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\beta_3,\epsilon,\epsilon}$, $\mathrm{F}^{\beta_4}_{\beta_3,\epsilon,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\epsilon,\alpha_+}$,
$\mathrm{F}^{\alpha_-}_{\beta_3,\epsilon,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\epsilon,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\epsilon,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_3,\epsilon,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\epsilon,\beta_3}$, $\mathrm{F}^{\epsilon}_{\beta_3,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_3,\epsilon,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_1,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_1,\mathrm{I}}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_1,\beta_1}$,
$\mathrm{F}^{\beta_4}_{\beta_3,\beta_1,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_1,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_1,\beta_4}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_1,\beta_4}$,
$\mathrm{F}^{\beta_1}_{\beta_3,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\alpha_+,\alpha_-}$,
$\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\mathrm{I}}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_2,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_2,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_2,\epsilon}$,
$\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_2,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_2,\beta_4}$,
$\mathrm{F}^{\beta_2}_{\beta_3,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_2,\beta_4}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_3,\mathrm{I}}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_3,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_3,\mathrm{I}}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_3,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_3,\epsilon}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_3,\beta_1}$,
$\mathrm{F}^{\beta_1}_{\beta_3,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_3,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_4,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_4,\beta_1}$,
$\mathrm{F}^{\epsilon}_{\beta_3,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_4,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_4,\beta_2}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_4,\beta_2}$,
$\mathrm{F}^{\beta_1}_{\beta_3,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_4,\mathrm{I},\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\beta_4,\mathrm{I},\epsilon}$, $\mathrm{F}^{\beta_2}_{\beta_4,\mathrm{I},\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_4,\mathrm{I},\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\mathrm{I},\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\mathrm{I},\alpha_+}$,
$\mathrm{F}^{\alpha_+}_{\beta_4,\mathrm{I},\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\mathrm{I},\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_4,\mathrm{I},\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_4,\mathrm{I},\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_4,\mathrm{I},\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_4,\mathrm{I},\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\mathrm{I},\beta_4}$, $\mathrm{F}^{\epsilon}_{\beta_4,\mathrm{I},\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_4,\mathrm{I},\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_4,\epsilon,\mathrm{I}}$,
$\mathrm{F}^{\beta_4}_{\beta_4,\epsilon,\epsilon}$, $\mathrm{F}^{\beta_2}_{\beta_4,\epsilon,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\epsilon,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\epsilon,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\epsilon,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\epsilon,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_4,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_4,\epsilon,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\epsilon,\beta_4}$, $\mathrm{F}^{\epsilon}_{\beta_4,\epsilon,\beta_4}$,
$\mathrm{F}^{\beta_2}_{\beta_4,\beta_1,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_1,\mathrm{I}}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_1,\epsilon}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_1,\epsilon}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_1,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_1,\beta_2}$,
$\mathrm{F}^{\beta_4}_{\beta_4,\beta_1,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_1,\beta_3}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\mathrm{I}}$,
$\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\mathrm{I}}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\alpha_-,\alpha_+}$,
$\mathrm{F}^{\epsilon}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_2,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_2,\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_2,\epsilon}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_2,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_2,\beta_1}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_2,\beta_1}$,
$\mathrm{F}^{\beta_2}_{\beta_4,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_2,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_2,\beta_4}$,
$\mathrm{F}^{\beta_3}_{\beta_4,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_3,\mathrm{I}}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_3,\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_3,\epsilon}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_3,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_3,\beta_1}$,
$\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_3,\beta_2}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_3,\beta_4}$,
$\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_4,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_4,\epsilon}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_4,\beta_3}$,
$\mathrm{F}^{\beta_2}_{\beta_4,\beta_4,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_4,\beta_4}$

A.3.2 Pivotal Structures

There are 2 pivotal structures.

- $\epsilon_{\mathrm{I}} = 1$, $\epsilon_{\epsilon} = 1$, $\epsilon_{\beta_1} = 1$, $\epsilon_{\alpha_+} = 1$, $\epsilon_{\alpha_-} = 1$, $\epsilon_{\beta_2} = 1$, $\epsilon_{\beta_3} = 1$, and $\epsilon_{\beta_4} = 1$ which is spherical with quantum dimensions $q_{\mathrm{I}} = 1$, $q_{\epsilon} = 1$, $q_{\beta_1} = 2$, $q_{\alpha_+} = 3$, $q_{\alpha_-} = 3$, $q_{\beta_2} = 2$, $q_{\beta_3} = 2$, and $q_{\beta_4} = 2$.

- $\epsilon_{\mathrm{I}} = 1$, $\epsilon_{\epsilon} = 1$, $\epsilon_{\beta_1} = 1$, $\epsilon_{\alpha_+} = -1$, $\epsilon_{\alpha_-} = -1$, $\epsilon_{\beta_2} = 1$, $\epsilon_{\beta_3} = 1$, and $\epsilon_{\beta_4} = 1$ which is spherical with quantum dimensions $q_{\mathrm{I}} = 1$, $q_{\epsilon} = 1$, $q_{\beta_1} = 2$, $q_{\alpha_+} = -3$, $q_{\alpha_-} = -3$, $q_{\beta_2} = 2$, $q_{\beta_3} = 2$, and $q_{\beta_4} = 2$.

A.3.3 Braidings

There is 1 braiding which corresponds to this solution.

- $\mathrm{R}^{\epsilon}_{\beta_3,\beta_3} = \mathrm{R}^{\epsilon}_{\beta_4,\beta_4} = \begin{pmatrix} \frac{1}{2}\left(1 - i\sqrt{3}\right) \end{pmatrix}$
- $R^{c}_{a,b} = \begin{pmatrix} -(-1)^{5/6} \end{pmatrix}$
  - $\mathrm{R}^{\beta_1}_{\alpha_-,\alpha_-}$, $\mathrm{R}^{\beta_2}_{\alpha_-,\alpha_-}$, $\mathrm{R}^{\alpha_+}_{\alpha_-,\beta_3}$, $\mathrm{R}^{\alpha_+}_{\alpha_-,\beta_4}$, $\mathrm{R}^{\alpha_+}_{\beta_3,\alpha_-}$, $\mathrm{R}^{\alpha_+}_{\beta_4,\alpha_-}$
- $R^{c}_{a,b} = \begin{pmatrix} (-1)^{5/6} \end{pmatrix}$
  - $\mathrm{R}^{\beta_1}_{\alpha_+,\alpha_+}$, $\mathrm{R}^{\beta_2}_{\alpha_+,\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\alpha_+,\beta_3}$, $\mathrm{R}^{\alpha_-}_{\alpha_+,\beta_4}$, $\mathrm{R}^{\alpha_-}_{\beta_3,\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\beta_4,\alpha_+}$
- $R^{c}_{a,b} = \begin{pmatrix} -\sqrt[6]{-1} \end{pmatrix}$
  - $\mathrm{R}^{\alpha_+}_{\beta_1,\alpha_-}$, $\mathrm{R}^{\alpha_+}_{\alpha_-,\beta_1}$, $\mathrm{R}^{\beta_3}_{\alpha_-,\alpha_-}$, $\mathrm{R}^{\beta_4}_{\alpha_-,\alpha_-}$, $\mathrm{R}^{\alpha_+}_{\alpha_-,\beta_2}$, $\mathrm{R}^{\alpha_+}_{\beta_2,\alpha_-}$
- $R^{c}_{a,b} = \begin{pmatrix} \sqrt[6]{-1} \end{pmatrix}$
  - $\mathrm{R}^{\alpha_-}_{\beta_1,\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\alpha_+,\beta_1}$, $\mathrm{R}^{\beta_3}_{\alpha_+,\alpha_+}$, $\mathrm{R}^{\beta_4}_{\alpha_+,\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\alpha_+,\beta_2}$, $\mathrm{R}^{\alpha_-}_{\beta_2,\alpha_+}$

- $R^c_{a,b} = \begin{pmatrix} \frac{1}{2}i\left(i+\sqrt{3}\right) \end{pmatrix}$
  - $\mathrm{R}^{\beta_1}_{\beta_1,\beta_1}, \mathrm{R}^{\alpha_+}_{\beta_1,\alpha_+}, \mathrm{R}^{\alpha_-}_{\beta_1,\alpha_-}, \mathrm{R}^{\beta_4}_{\beta_1,\beta_3}, \mathrm{R}^{\beta_3}_{\beta_1,\beta_4}, \mathrm{R}^{\alpha_+}_{\alpha_+,\beta_1}, \mathrm{R}^{\beta_3}_{\alpha_+,\alpha_-}, \mathrm{R}^{\beta_4}_{\alpha_+,\alpha_-}, \mathrm{R}^{\alpha_+}_{\alpha_+,\beta_2}, \mathrm{R}^{\alpha_-}_{\alpha_-,\beta_1}, \mathrm{R}^{\beta_3}_{\alpha_-,\alpha_+}, \mathrm{R}^{\beta_4}_{\alpha_-,\alpha_+}, \mathrm{R}^{\alpha_-}_{\alpha_-,\beta_2}, \mathrm{R}^{\alpha_+}_{\beta_2,\alpha_+}, \mathrm{R}^{\alpha_-}_{\beta_2,\alpha_-}, \mathrm{R}^{\beta_2}_{\beta_2,\beta_2}, \mathrm{R}^{\beta_4}_{\beta_2,\beta_3}, \mathrm{R}^{\beta_3}_{\beta_2,\beta_4}, \mathrm{R}^{\beta_4}_{\beta_3,\beta_1}, \mathrm{R}^{\beta_4}_{\beta_3,\beta_2}, \mathrm{R}^{\mathrm{I}}_{\beta_3,\beta_3}, \mathrm{R}^{\beta_3}_{\beta_4,\beta_1}, \mathrm{R}^{\beta_3}_{\beta_4,\beta_2}, \mathrm{R}^{\mathrm{I}}_{\beta_4,\beta_4}$
- $\mathrm{R}^{\epsilon}_{\beta_1,\beta_1} = \mathrm{R}^{\epsilon}_{\beta_2,\beta_2} = \begin{pmatrix} \frac{1}{2}\left(1+i\sqrt{3}\right) \end{pmatrix}$
- $R^c_{a,b} = \begin{pmatrix} -\frac{1}{2}i\left(-i+\sqrt{3}\right) \end{pmatrix}$
  - $\mathrm{R}^{\mathrm{I}}_{\beta_1,\beta_1}, \mathrm{R}^{\beta_2}_{\beta_1,\beta_3}, \mathrm{R}^{\beta_2}_{\beta_1,\beta_4}, \mathrm{R}^{\beta_1}_{\alpha_+,\alpha_-}, \mathrm{R}^{\beta_2}_{\alpha_+,\alpha_-}, \mathrm{R}^{\alpha_+}_{\alpha_+,\beta_3}, \mathrm{R}^{\alpha_+}_{\alpha_+,\beta_4}, \mathrm{R}^{\beta_1}_{\alpha_-,\alpha_+}, \mathrm{R}^{\beta_2}_{\alpha_-,\alpha_+}, \mathrm{R}^{\alpha_-}_{\alpha_-,\beta_3}, \mathrm{R}^{\alpha_-}_{\alpha_-,\beta_4}, \mathrm{R}^{\mathrm{I}}_{\beta_2,\beta_2}, \mathrm{R}^{\beta_1}_{\beta_2,\beta_3}, \mathrm{R}^{\beta_1}_{\beta_2,\beta_4}, \mathrm{R}^{\beta_2}_{\beta_3,\beta_1}, \mathrm{R}^{\alpha_+}_{\beta_3,\alpha_+}, \mathrm{R}^{\alpha_-}_{\beta_3,\alpha_-}, \mathrm{R}^{\beta_1}_{\beta_3,\beta_2}, \mathrm{R}^{\beta_3}_{\beta_3,\beta_3}, \mathrm{R}^{\beta_2}_{\beta_4,\beta_1}, \mathrm{R}^{\alpha_+}_{\beta_4,\alpha_+}, \mathrm{R}^{\alpha_-}_{\beta_4,\alpha_-}, \mathrm{R}^{\beta_1}_{\beta_4,\beta_2}, \mathrm{R}^{\beta_4}_{\beta_4,\beta_4}$
- $\mathrm{R}^{\alpha_+}_{\epsilon,\alpha_-} = \mathrm{R}^{\mathrm{I}}_{\alpha_+,\alpha_+} = \mathrm{R}^{\alpha_+}_{\alpha_-,\epsilon} = \begin{pmatrix} -i \end{pmatrix}$
- $\mathrm{R}^{\alpha_-}_{\epsilon,\alpha_+} = \mathrm{R}^{\alpha_-}_{\alpha_+,\epsilon} = \mathrm{R}^{\mathrm{I}}_{\alpha_-,\alpha_-} = \begin{pmatrix} i \end{pmatrix}$
- $R^c_{a,b} = \begin{pmatrix} -1 \end{pmatrix}$
  - $\mathrm{R}^{\beta_1}_{\epsilon,\beta_1}, \mathrm{R}^{\beta_2}_{\epsilon,\beta_2}, \mathrm{R}^{\beta_3}_{\epsilon,\beta_3}, \mathrm{R}^{\beta_4}_{\epsilon,\beta_4}, \mathrm{R}^{\beta_1}_{\beta_1,\epsilon}, \mathrm{R}^{\epsilon}_{\alpha_+,\alpha_-}, \mathrm{R}^{\epsilon}_{\alpha_-,\alpha_+}, \mathrm{R}^{\beta_2}_{\beta_2,\epsilon}, \mathrm{R}^{\beta_3}_{\beta_3,\epsilon}, \mathrm{R}^{\beta_4}_{\beta_4,\epsilon}$
- $R^c_{a,b} = \begin{pmatrix} 1 \end{pmatrix}$
  - $\mathrm{R}^{\mathrm{I}}_{\mathrm{I},\mathrm{I}}, \mathrm{R}^{\epsilon}_{\mathrm{I},\epsilon}, \mathrm{R}^{\beta_1}_{\mathrm{I},\beta_1}, \mathrm{R}^{\alpha_+}_{\mathrm{I},\alpha_+}, \mathrm{R}^{\alpha_-}_{\mathrm{I},\alpha_-}, \mathrm{R}^{\beta_2}_{\mathrm{I},\beta_2}, \mathrm{R}^{\beta_3}_{\mathrm{I},\beta_3}, \mathrm{R}^{\beta_4}_{\mathrm{I},\beta_4}, \mathrm{R}^{\epsilon}_{\epsilon,\mathrm{I}}, \mathrm{R}^{\mathrm{I}}_{\epsilon,\epsilon}, \mathrm{R}^{\beta_1}_{\beta_1,\mathrm{I}}, \mathrm{R}^{\beta_3}_{\beta_1,\beta_2}, \mathrm{R}^{\beta_4}_{\beta_1,\beta_2}, \mathrm{R}^{\alpha_+}_{\alpha_+,\mathrm{I}}, \mathrm{R}^{\alpha_-}_{\alpha_-,\mathrm{I}}, \mathrm{R}^{\beta_2}_{\beta_2,\mathrm{I}}, \mathrm{R}^{\beta_3}_{\beta_2,\beta_1}, \mathrm{R}^{\beta_4}_{\beta_2,\beta_1}, \mathrm{R}^{\beta_3}_{\beta_3,\mathrm{I}}, \mathrm{R}^{\beta_1}_{\beta_3,\beta_4}, \mathrm{R}^{\beta_2}_{\beta_3,\beta_4}, \mathrm{R}^{\beta_4}_{\beta_4,\mathrm{I}}, \mathrm{R}^{\beta_1}_{\beta_4,\beta_3}, \mathrm{R}^{\beta_2}_{\beta_4,\beta_3}$

A.3.4 Balanced, Ribbon, and Modular Structures

There are 2 balanced categories which correspond to this solution. Both of them are modular.

Balanced Category 1 has pivotal structure 1. Its T-Matrix is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) \end{pmatrix}.$$

It is ribbon and has S-Matrix given by:

$$\begin{pmatrix} 1 & 1 & 2 & 3 & 3 & 2 & 2 & 2 \\ 1 & 1 & 2 & -3 & -3 & 2 & 2 & 2 \\ 2 & 2 & -2 & 0 & 0 & 4 & -2 & -2 \\ 3 & -3 & 0 & 3 & -3 & 0 & 0 & 0 \\ 3 & -3 & 0 & -3 & 3 & 0 & 0 & 0 \\ 2 & 2 & 4 & 0 & 0 & -2 & -2 & -2 \\ 2 & 2 & -2 & 0 & 0 & -2 & -2 & 4 \\ 2 & 2 & -2 & 0 & 0 & -2 & 4 & -2 \end{pmatrix}.$$

Balanced Category 2 has pivotal structure 2. Its T-Matrix is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) \end{pmatrix}.$$

It is ribbon and has S-Matrix given by:

$$\begin{pmatrix} 1 & 1 & 2 & -3 & -3 & 2 & 2 & 2 \\ 1 & 1 & 2 & 3 & 3 & 2 & 2 & 2 \\ 2 & 2 & -2 & 0 & 0 & 4 & -2 & -2 \\ -3 & 3 & 0 & 3 & -3 & 0 & 0 & 0 \\ -3 & 3 & 0 & -3 & 3 & 0 & 0 & 0 \\ 2 & 2 & 4 & 0 & 0 & -2 & -2 & -2 \\ 2 & 2 & -2 & 0 & 0 & -2 & -2 & 4 \\ 2 & 2 & -2 & 0 & 0 & -2 & 4 & -2 \end{pmatrix}.$$

A.4 Solution 4 – Non-Group Theoretical Category 2

A.4.1 F-Matrices

- $\mathrm{F}^{\alpha_+}_{\alpha_+,\alpha_-,\alpha_-} = \mathrm{F}^{\alpha_+}_{\alpha_-,\alpha_-,\alpha_+} = \mathrm{F}^{\alpha_-}_{\alpha_-,\alpha_-,\alpha_-} = \begin{pmatrix} -\frac{1}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{\sqrt{2}}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$

- $\mathrm{F}^{\alpha_-}_{\alpha_+,\alpha_-,\alpha_+} = \begin{pmatrix} \frac{1}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{\sqrt{2}}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$

- $\mathrm{F}^{\alpha_-}_{\alpha_+,\alpha_+,\alpha_-} = \mathrm{F}^{\alpha_-}_{\alpha_-,\alpha_+,\alpha_+} = \mathrm{F}^{\alpha_+}_{\alpha_-,\alpha_+,\alpha_-} = \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{\sqrt{2}}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{\sqrt{2}}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$

- $\mathrm{F}^{\alpha_+}_{\alpha_+,\alpha_+,\alpha_+} = \begin{pmatrix} -\frac{1}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{pmatrix}$

- $\mathrm{F}^{\alpha_-}_{\alpha_-,\alpha_+,\alpha_-} = \mathrm{F}^{\alpha_+}_{\alpha_-,\alpha_-,\alpha_-} = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \end{pmatrix}$
- $F^d_{abc} = \begin{pmatrix} -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_-}_{\alpha_+,\alpha_+,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\alpha_+,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\alpha_-,\alpha_+}$
- $\mathrm{F}^{\beta_1}_{\beta_1,\beta_1,\beta_1} = \mathrm{F}^{\beta_2}_{\beta_2,\beta_2,\beta_2} = \mathrm{F}^{\beta_3}_{\beta_3,\beta_3,\beta_3} = \mathrm{F}^{\beta_4}_{\beta_4,\beta_4,\beta_4} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$
- $F^d_{abc} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$
  - $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_4,\alpha_+}$
- $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_1,\alpha_+} = \mathrm{F}^{\beta_1}_{\alpha_-,\beta_2,\alpha_+} = \mathrm{F}^{\beta_4}_{\alpha_-,\beta_3,\alpha_+} = \mathrm{F}^{\beta_3}_{\alpha_-,\beta_4,\alpha_+} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_1,\alpha_+} = \mathrm{F}^{\beta_4}_{\alpha_-,\beta_1,\alpha_+} = \mathrm{F}^{\beta_2}_{\alpha_-,\beta_2,\alpha_+} = \mathrm{F}^{\beta_1}_{\alpha_-,\beta_4,\alpha_+} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$
- $F^d_{abc} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$
  - $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_4,\alpha_-}$

- $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_1,\alpha_-} = \mathrm{F}^{\beta_1}_{\alpha_+,\beta_2,\alpha_-} = \mathrm{F}^{\beta_4}_{\alpha_+,\beta_3,\alpha_-} = \mathrm{F}^{\beta_3}_{\alpha_+,\beta_4,\alpha_-} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
- $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_1,\alpha_-} = \mathrm{F}^{\beta_4}_{\alpha_+,\beta_1,\alpha_-} = \mathrm{F}^{\beta_2}_{\alpha_+,\beta_2,\alpha_-} = \mathrm{F}^{\beta_1}_{\alpha_+,\beta_4,\alpha_-} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$
- $F^d_{abc} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\beta_4}$
- $F^d_{abc} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
  - $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\beta_3}$
- $F^d_{abc} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\beta_1}$
- $F^d_{abc} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\beta_4}$
- $F^d_{abc} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

- $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\beta_4}$

- $F^d_{abc} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

  - $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_1,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_2,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_2,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_1,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_3,\beta_4}$

- $F^d_{abc} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

  - $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\beta_3}$

- $F^d_{abc} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

  - $\mathrm{F}^{\beta_2}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_4,\alpha_-,\alpha_-}$

- $F^d_{abc} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

  - $\mathrm{F}^{\beta_1}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_2,\alpha_-}$

- $F^d_{abc} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\beta_1}$
- $F^d_{abc} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\beta_1}$
- $F^d_{abc} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$
  - $\mathrm{F}^{\beta_4}_{\beta_1,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_4,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\beta_1}$,
    $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\beta_4}$,
    $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_1,\alpha_+}$,
    $\mathrm{F}^{\beta_1}_{\beta_2,\alpha_+,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_2,\alpha_-}$,
    $\mathrm{F}^{\beta_4}_{\beta_3,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_2,\alpha_-}$,
    $\mathrm{F}^{\beta_4}_{\beta_4,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_4,\beta_3}$
- $F^d_{abc} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$
  - $\mathrm{F}^{\beta_2}_{\beta_1,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\beta_1}$,
    $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\beta_1}$,
    $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_1,\alpha_-}$,
    $\mathrm{F}^{\beta_4}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_2,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_2,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_1,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_2,\alpha_+}$,
    $\mathrm{F}^{\beta_1}_{\beta_3,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_4,\beta_2}$
- $F^d_{abc} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

- $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_4,\alpha_+}$

- $F^d_{abc} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

  - $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_2,\alpha_+,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_4,\alpha_+}$

- $F^d_{abc} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

  - $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_4,\alpha_-}$

- $F^d_{abc} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

  - $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_2,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_2,\alpha_+,\alpha_+}$,

$\mathrm{F}^{\beta_4}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_2}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_4,\alpha_-}$

- $F^d_{abc} = \begin{pmatrix} -1 \end{pmatrix}$

  - $\mathrm{F}^{\beta_1}_{\epsilon,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_1,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_1,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\beta_1}$, $\mathrm{F}^{\epsilon}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\epsilon,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\epsilon,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\epsilon,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\epsilon,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\epsilon,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\epsilon,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\epsilon,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\epsilon,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_3,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_1,\epsilon,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_1,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_1,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_3}_{\beta_1,\epsilon,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_1,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_2,\epsilon}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_2,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_3,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_3,\beta_2}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_4,\epsilon}$, $\mathrm{F}^{\epsilon}_{\alpha_+,\epsilon,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\alpha_+,\epsilon,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\epsilon}$, $\mathrm{F}^{\epsilon}_{\alpha_+,\alpha_+,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_-,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\alpha_-,\epsilon,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\alpha_-,\epsilon,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\epsilon}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\epsilon}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\epsilon}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\epsilon}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\epsilon}$, $\mathrm{F}^{\epsilon}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\epsilon}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\epsilon}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\epsilon}_{\alpha_-,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\epsilon}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\epsilon}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\epsilon}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\epsilon}$, $\mathrm{F}^{\beta_3}_{\beta_2,\epsilon,\beta_1}$, $\mathrm{F}^{\beta_2}_{\beta_2,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_2,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_2,\epsilon,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_2,\beta_1,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_1,\beta_3}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_2,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_3,\epsilon}$, $\mathrm{F}^{\beta_4}_{\beta_2,\beta_3,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_2,\beta_4,\epsilon}$, $\mathrm{F}^{\beta_2}_{\beta_3,\epsilon,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_3,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_3,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_3,\epsilon,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_1,\epsilon}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_1,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_1,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_2,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_2,\beta_1}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_3,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_4,\epsilon}$, $\mathrm{F}^{\beta_3}_{\beta_4,\epsilon,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_4,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_4,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_4,\epsilon,\beta_4}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_1,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_2,\beta_3}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_4,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_4,\beta_4}$

- $F^d_{abc} = \begin{pmatrix} 1 \end{pmatrix}$

  - $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\mathrm{I},\mathrm{I}}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\mathrm{I},\epsilon}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\mathrm{I},\beta_1}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\mathrm{I},\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\mathrm{I},\alpha_-}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\mathrm{I},\beta_2}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\mathrm{I},\beta_3}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\mathrm{I},\beta_4}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\epsilon,\mathrm{I}}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\epsilon,\epsilon}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\epsilon,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\epsilon,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\epsilon,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\epsilon,\beta_2}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\epsilon,\beta_3}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\epsilon,\beta_4}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_1,\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_1,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\beta_1,\beta_1}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\beta_1,\beta_1}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_1,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_1,\beta_2}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_1,\beta_2}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_1,\beta_3}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_1,\beta_3}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\epsilon}$,

$\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\alpha_+,\alpha_-}$,
$\mathrm{F}^{\beta_2}_{\mathrm{I},\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\beta_4}$,
$\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_-,\mathrm{I}}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_-,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_-,\beta_1}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\alpha_-,\alpha_+}$,
$\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_-,\beta_3}$,
$\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_2,\mathrm{I}}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_2,\epsilon}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_2,\beta_1}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_2,\alpha_-}$,
$\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\beta_2,\beta_2}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\beta_2,\beta_2}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_2,\beta_2}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_2,\beta_3}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_2,\beta_3}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_3,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_3,\epsilon}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_3,\beta_1}$,
$\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_3,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_3,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\beta_3,\beta_3}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\beta_3,\beta_3}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_3,\beta_3}$,
$\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_3,\beta_4}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_3,\beta_4}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_4,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_4,\epsilon}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_4,\beta_1}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_4,\alpha_-}$,
$\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_4,\beta_2}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_4,\beta_2}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_4,\beta_3}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_4,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\beta_4,\beta_4}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\beta_4,\beta_4}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_4,\beta_4}$, $\mathrm{F}^{\epsilon}_{\epsilon,\mathrm{I},\mathrm{I}}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\mathrm{I},\epsilon}$, $\mathrm{F}^{\beta_1}_{\epsilon,\mathrm{I},\beta_1}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\mathrm{I},\alpha_+}$,
$\mathrm{F}^{\alpha_+}_{\epsilon,\mathrm{I},\alpha_-}$, $\mathrm{F}^{\beta_2}_{\epsilon,\mathrm{I},\beta_2}$, $\mathrm{F}^{\beta_3}_{\epsilon,\mathrm{I},\beta_3}$, $\mathrm{F}^{\beta_4}_{\epsilon,\mathrm{I},\beta_4}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\epsilon,\mathrm{I}}$, $\mathrm{F}^{\epsilon}_{\epsilon,\epsilon,\epsilon}$, $\mathrm{F}^{\beta_1}_{\epsilon,\epsilon,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\epsilon,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\epsilon,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\epsilon,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_3}_{\epsilon,\epsilon,\beta_3}$,
$\mathrm{F}^{\beta_4}_{\epsilon,\epsilon,\beta_4}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_1,\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_1,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\beta_1,\beta_1}$, $\mathrm{F}^{\epsilon}_{\epsilon,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_+,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_+,\beta_1}$,
$\mathrm{F}^{\beta_1}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_-,\mathrm{I}}$,
$\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_1}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\epsilon,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_-,\beta_4}$,
$\mathrm{F}^{\beta_2}_{\epsilon,\beta_2,\mathrm{I}}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_2,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\beta_2,\beta_2}$, $\mathrm{F}^{\epsilon}_{\epsilon,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_3,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_3,\epsilon}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_3,\beta_1}$,
$\mathrm{F}^{\mathrm{I}}_{\epsilon,\beta_3,\beta_3}$, $\mathrm{F}^{\epsilon}_{\epsilon,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_4,\epsilon}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_4,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\beta_4,\beta_4}$,
$\mathrm{F}^{\epsilon}_{\epsilon,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_1,\mathrm{I},\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\beta_1,\mathrm{I},\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\mathrm{I},\beta_1}$, $\mathrm{F}^{\epsilon}_{\beta_1,\mathrm{I},\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_1,\mathrm{I},\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\mathrm{I},\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\mathrm{I},\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\mathrm{I},\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\mathrm{I},\alpha_-}$,
$\mathrm{F}^{\beta_3}_{\beta_1,\mathrm{I},\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_1,\mathrm{I},\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_1,\mathrm{I},\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_1,\mathrm{I},\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_1,\mathrm{I},\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_1,\mathrm{I},\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_1,\epsilon,\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\beta_1,\epsilon,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\epsilon,\beta_1}$, $\mathrm{F}^{\epsilon}_{\beta_1,\epsilon,\beta_1}$,
$\mathrm{F}^{\alpha_+}_{\beta_1,\epsilon,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\epsilon,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\epsilon,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\epsilon,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_1,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_1,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_1,\epsilon,\beta_4}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_1,\mathrm{I}}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_1,\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_1,\mathrm{I}}$,
$\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_1,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_1,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_1,\beta_4}$,
$\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\mathrm{I}}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\epsilon}$,
$\mathrm{F}^{\mathrm{I}}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_2,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_2,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_2,\beta_2}$,
$\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_2,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_2,\beta_4}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_2,\beta_4}$,
$\mathrm{F}^{\beta_2}_{\beta_1,\beta_3,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_3,\mathrm{I}}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_3,\epsilon}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_3,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_3,\beta_2}$,
$\mathrm{F}^{\beta_2}_{\beta_1,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_3,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_4,\mathrm{I}}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_4,\epsilon}$,
$\mathrm{F}^{\beta_2}_{\beta_1,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_4,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_4,\beta_2}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_4,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\beta_1,\beta_4,\beta_3}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_4,\beta_3}$,
$\mathrm{F}^{\beta_1}_{\beta_1,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_4,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\mathrm{I},\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\mathrm{I},\epsilon}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\mathrm{I},\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\mathrm{I},\beta_1}$,

$$\mathrm{F}^{\mathrm{I}}_{\alpha_+,\mathrm{I},\alpha_+},\ \mathrm{F}^{\beta_1}_{\alpha_+,\mathrm{I},\alpha_+},\ \mathrm{F}^{\beta_2}_{\alpha_+,\mathrm{I},\alpha_+},\ \mathrm{F}^{\beta_3}_{\alpha_+,\mathrm{I},\alpha_+},\ \mathrm{F}^{\beta_4}_{\alpha_+,\mathrm{I},\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_+,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_1}_{\alpha_+,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_2}_{\alpha_+,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_3}_{\alpha_+,\mathrm{I},\alpha_-},$$

$$\mathrm{F}^{\beta_4}_{\alpha_+,\mathrm{I},\alpha_-},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\mathrm{I},\beta_2},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\mathrm{I},\beta_2},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\mathrm{I},\beta_3},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\mathrm{I},\beta_3},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\mathrm{I},\beta_4},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\mathrm{I},\beta_4},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\epsilon,\mathrm{I}},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\epsilon,\epsilon},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\epsilon,\beta_1},$$

$$\mathrm{F}^{\alpha_-}_{\alpha_+,\epsilon,\beta_1},\ \mathrm{F}^{\beta_1}_{\alpha_+,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_2}_{\alpha_+,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_3}_{\alpha_+,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_4}_{\alpha_+,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_1}_{\alpha_+,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_2}_{\alpha_+,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_3}_{\alpha_+,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_4}_{\alpha_+,\epsilon,\alpha_-},$$

$$\mathrm{F}^{\alpha_+}_{\alpha_+,\epsilon,\beta_2},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\epsilon,\beta_2},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\epsilon,\beta_3},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\epsilon,\beta_3},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\epsilon,\beta_4},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\epsilon,\beta_4},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_1,\alpha_+},$$

$$\mathrm{F}^{\epsilon}_{\alpha_+,\beta_1,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_1,\alpha_-},\ \mathrm{F}^{\epsilon}_{\alpha_+,\beta_1,\alpha_-},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\epsilon},$$

$$\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\epsilon},\ \mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\epsilon},\ \mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_+,\beta_1},\ \mathrm{F}^{\epsilon}_{\alpha_+,\alpha_+,\beta_1},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_+,\beta_2},\ \mathrm{F}^{\epsilon}_{\alpha_+,\alpha_+,\beta_2},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_+,\beta_3},$$

$$\mathrm{F}^{\epsilon}_{\alpha_+,\alpha_+,\beta_3},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_+,\beta_4},\ \mathrm{F}^{\epsilon}_{\alpha_+,\alpha_+,\beta_4},\ \mathrm{F}^{\epsilon}_{\alpha_+,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\epsilon},$$

$$\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\epsilon},\ \mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\epsilon},\ \mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_-,\beta_1},\ \mathrm{F}^{\epsilon}_{\alpha_+,\alpha_-,\beta_1},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_-,\beta_2},\ \mathrm{F}^{\epsilon}_{\alpha_+,\alpha_-,\beta_2},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_-,\beta_3},$$

$$\mathrm{F}^{\epsilon}_{\alpha_+,\alpha_-,\beta_3},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_-,\beta_4},\ \mathrm{F}^{\epsilon}_{\alpha_+,\alpha_-,\beta_4},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_2,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_+,\beta_2,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_2,\alpha_-},$$

$$\mathrm{F}^{\epsilon}_{\alpha_+,\beta_2,\alpha_-},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_3,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_+,\beta_3,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_3,\alpha_-},\ \mathrm{F}^{\epsilon}_{\alpha_+,\beta_3,\alpha_-},\ \mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\mathrm{I}},$$

$$\mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_4,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_+,\beta_4,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\alpha_+,\beta_4,\alpha_-},\ \mathrm{F}^{\epsilon}_{\alpha_+,\beta_4,\alpha_-},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\mathrm{I},\mathrm{I}},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\mathrm{I},\epsilon},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\mathrm{I},\beta_1},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\mathrm{I},\beta_1},\ \mathrm{F}^{\epsilon}_{\alpha_-,\mathrm{I},\alpha_+},$$

$$\mathrm{F}^{\beta_1}_{\alpha_-,\mathrm{I},\alpha_+},\ \mathrm{F}^{\beta_2}_{\alpha_-,\mathrm{I},\alpha_+},\ \mathrm{F}^{\beta_3}_{\alpha_-,\mathrm{I},\alpha_+},\ \mathrm{F}^{\beta_4}_{\alpha_-,\mathrm{I},\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_1}_{\alpha_-,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_2}_{\alpha_-,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_3}_{\alpha_-,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_4}_{\alpha_-,\mathrm{I},\alpha_-},$$

$$\mathrm{F}^{\alpha_+}_{\alpha_-,\mathrm{I},\beta_2},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\mathrm{I},\beta_2},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\mathrm{I},\beta_3},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\mathrm{I},\beta_3},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\mathrm{I},\beta_4},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\mathrm{I},\beta_4},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\epsilon,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\epsilon,\epsilon},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\epsilon,\beta_1},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\epsilon,\beta_1},$$

$$\mathrm{F}^{\beta_1}_{\alpha_-,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_2}_{\alpha_-,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_3}_{\alpha_-,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_4}_{\alpha_-,\epsilon,\alpha_+},\ \mathrm{F}^{\beta_1}_{\alpha_-,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_2}_{\alpha_-,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_3}_{\alpha_-,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_4}_{\alpha_-,\epsilon,\alpha_-},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\epsilon,\beta_2},$$

$$\mathrm{F}^{\alpha_-}_{\alpha_-,\epsilon,\beta_2},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\epsilon,\beta_3},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\epsilon,\beta_3},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\epsilon,\beta_4},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\epsilon,\beta_4},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_1,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_1,\alpha_+},$$

$$\mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_1,\alpha_-},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_1,\alpha_-},\ \mathrm{F}^{\epsilon}_{\alpha_-,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_+,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_+,\beta_1},$$

$$\mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_+,\beta_2},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_+,\beta_3},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_+,\beta_4},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\epsilon}_{\alpha_-,\alpha_-,\epsilon},$$

$$\mathrm{F}^{\epsilon}_{\alpha_-,\alpha_-,\beta_1},\ \mathrm{F}^{\epsilon}_{\alpha_-,\alpha_-,\beta_2},\ \mathrm{F}^{\epsilon}_{\alpha_-,\alpha_-,\beta_3},\ \mathrm{F}^{\epsilon}_{\alpha_-,\alpha_-,\beta_4},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_2,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_2,\alpha_+},$$

$$\mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_2,\alpha_-},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_2,\alpha_-},\ \mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_3,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_3,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_3,\alpha_-},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_3,\alpha_-},$$

$$\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_4,\alpha_+},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_4,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\alpha_-,\beta_4,\alpha_-},\ \mathrm{F}^{\epsilon}_{\alpha_-,\beta_4,\alpha_-},\ \mathrm{F}^{\beta_2}_{\beta_2,\mathrm{I},\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\beta_2,\mathrm{I},\epsilon},\ \mathrm{F}^{\beta_3}_{\beta_2,\mathrm{I},\beta_1},$$

$$\mathrm{F}^{\beta_4}_{\beta_2,\mathrm{I},\beta_1},\ \mathrm{F}^{\alpha_+}_{\beta_2,\mathrm{I},\alpha_+},\ \mathrm{F}^{\alpha_-}_{\beta_2,\mathrm{I},\alpha_+},\ \mathrm{F}^{\alpha_+}_{\beta_2,\mathrm{I},\alpha_-},\ \mathrm{F}^{\alpha_-}_{\beta_2,\mathrm{I},\alpha_-},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\mathrm{I},\beta_2},\ \mathrm{F}^{\epsilon}_{\beta_2,\mathrm{I},\beta_2},\ \mathrm{F}^{\beta_2}_{\beta_2,\mathrm{I},\beta_2},\ \mathrm{F}^{\beta_1}_{\beta_2,\mathrm{I},\beta_3},\ \mathrm{F}^{\beta_4}_{\beta_2,\mathrm{I},\beta_3},$$

$$\mathrm{F}^{\beta_1}_{\beta_2,\mathrm{I},\beta_4},\ \mathrm{F}^{\beta_3}_{\beta_2,\mathrm{I},\beta_4},\ \mathrm{F}^{\beta_2}_{\beta_2,\epsilon,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\beta_2,\epsilon,\epsilon},\ \mathrm{F}^{\beta_4}_{\beta_2,\epsilon,\beta_1},\ \mathrm{F}^{\alpha_+}_{\beta_2,\epsilon,\alpha_+},\ \mathrm{F}^{\alpha_-}_{\beta_2,\epsilon,\alpha_+},\ \mathrm{F}^{\alpha_+}_{\beta_2,\epsilon,\alpha_-},\ \mathrm{F}^{\alpha_-}_{\beta_2,\epsilon,\alpha_-},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\epsilon,\beta_2},$$

$$\mathrm{F}^{\epsilon}_{\beta_2,\epsilon,\beta_2},\ \mathrm{F}^{\beta_4}_{\beta_2,\epsilon,\beta_3},\ \mathrm{F}^{\beta_3}_{\beta_2,\epsilon,\beta_4},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_1,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_1,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_1,\epsilon},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_1,\beta_1},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_1,\beta_1},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_1,\beta_2},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_1,\beta_2},$$

$$\mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_1,\beta_3},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_1,\beta_3},\ \mathrm{F}^{\beta_2}_{\beta_2,\beta_1,\beta_3},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_1,\beta_3},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_1,\beta_4},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_1,\beta_4},\ \mathrm{F}^{\beta_2}_{\beta_2,\beta_1,\beta_4},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_1,\beta_4},\ \mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\mathrm{I}},$$

$$\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\alpha_+,\alpha_+},\ \mathrm{F}^{\epsilon}_{\beta_2,\alpha_+,\alpha_+},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\alpha_+,\alpha_-},\ \mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\mathrm{I}},\ \mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\alpha_-,\alpha_+},$$

$$\mathrm{F}^{\epsilon}_{\beta_2,\alpha_-,\alpha_+},\ \mathrm{F}^{\epsilon}_{\beta_2,\alpha_-,\alpha_-},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_2,\mathrm{I}},\ \mathrm{F}^{\epsilon}_{\beta_2,\beta_2,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\beta_2,\beta_2,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_2,\epsilon},\ \mathrm{F}^{\epsilon}_{\beta_2,\beta_2,\epsilon},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_2,\beta_1},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_2,\beta_1},$$

$$\mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_2,\beta_2},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_2,\beta_3},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_2,\beta_3},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_2,\beta_4},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_2,\beta_4},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_3,\mathrm{I}},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_3,\mathrm{I}},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_3,\beta_1},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_3,\beta_1},$$
$$\mathrm{F}^{\beta_2}_{\beta_2,\beta_3,\beta_1},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_3,\beta_1},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_3,\beta_2},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_3,\beta_2},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_3,\beta_3},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_3,\beta_3},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_3,\beta_4},\ \mathrm{F}^{\epsilon}_{\beta_2,\beta_3,\beta_4},\ \mathrm{F}^{\beta_2}_{\beta_2,\beta_3,\beta_4},$$
$$\mathrm{F}^{\beta_3}_{\beta_2,\beta_3,\beta_4},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_3,\beta_4},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_4,\mathrm{I}},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_4,\mathrm{I}},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_4,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_4,\beta_1},\ \mathrm{F}^{\epsilon}_{\beta_2,\beta_4,\beta_1},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_4,\beta_1},\ \mathrm{F}^{\beta_2}_{\beta_2,\beta_4,\beta_1},$$
$$\mathrm{F}^{\beta_4}_{\beta_2,\beta_4,\beta_1},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_4,\beta_2},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_4,\beta_2},\ \mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_4,\beta_3},\ \mathrm{F}^{\epsilon}_{\beta_2,\beta_4,\beta_3},\ \mathrm{F}^{\beta_2}_{\beta_2,\beta_4,\beta_3},\ \mathrm{F}^{\beta_3}_{\beta_2,\beta_4,\beta_3},\ \mathrm{F}^{\beta_4}_{\beta_2,\beta_4,\beta_3},\ \mathrm{F}^{\beta_1}_{\beta_2,\beta_4,\beta_4},$$
$$\mathrm{F}^{\beta_3}_{\beta_2,\beta_4,\beta_4},\ \mathrm{F}^{\beta_3}_{\beta_3,\mathrm{I},\mathrm{I}},\ \mathrm{F}^{\beta_3}_{\beta_3,\mathrm{I},\epsilon},\ \mathrm{F}^{\beta_2}_{\beta_3,\mathrm{I},\beta_1},\ \mathrm{F}^{\beta_4}_{\beta_3,\mathrm{I},\beta_1},\ \mathrm{F}^{\alpha_+}_{\beta_3,\mathrm{I},\alpha_+},\ \mathrm{F}^{\alpha_-}_{\beta_3,\mathrm{I},\alpha_+},\ \mathrm{F}^{\alpha_+}_{\beta_3,\mathrm{I},\alpha_-},\ \mathrm{F}^{\alpha_-}_{\beta_3,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_1}_{\beta_3,\mathrm{I},\beta_2},$$
$$\mathrm{F}^{\beta_4}_{\beta_3,\mathrm{I},\beta_2},\ \mathrm{F}^{\mathrm{I}}_{\beta_3,\mathrm{I},\beta_3},\ \mathrm{F}^{\epsilon}_{\beta_3,\mathrm{I},\beta_3},\ \mathrm{F}^{\beta_3}_{\beta_3,\mathrm{I},\beta_3},\ \mathrm{F}^{\beta_1}_{\beta_3,\mathrm{I},\beta_4},\ \mathrm{F}^{\beta_2}_{\beta_3,\mathrm{I},\beta_4},\ \mathrm{F}^{\beta_3}_{\beta_3,\epsilon,\mathrm{I}},\ \mathrm{F}^{\beta_3}_{\beta_3,\epsilon,\epsilon},\ \mathrm{F}^{\beta_4}_{\beta_3,\epsilon,\beta_1},\ \mathrm{F}^{\alpha_+}_{\beta_3,\epsilon,\alpha_+},$$
$$\mathrm{F}^{\alpha_-}_{\beta_3,\epsilon,\alpha_+},\mathrm{F}^{\alpha_+}_{\beta_3,\epsilon,\alpha_-},\mathrm{F}^{\alpha_-}_{\beta_3,\epsilon,\alpha_-},\mathrm{F}^{\beta_4}_{\beta_3,\epsilon,\beta_2},\mathrm{F}^{\mathrm{I}}_{\beta_3,\epsilon,\beta_3},\mathrm{F}^{\epsilon}_{\beta_3,\epsilon,\beta_3},\mathrm{F}^{\beta_1}_{\beta_3,\epsilon,\beta_4},\mathrm{F}^{\beta_2}_{\beta_3,\beta_1,\mathrm{I}},\mathrm{F}^{\beta_4}_{\beta_3,\beta_1,\mathrm{I}},\mathrm{F}^{\beta_2}_{\beta_3,\beta_1,\beta_1},$$
$$\mathrm{F}^{\beta_4}_{\beta_3,\beta_1,\beta_1},\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_1,\beta_2},\mathrm{F}^{\beta_1}_{\beta_3,\beta_1,\beta_2},\mathrm{F}^{\beta_2}_{\beta_3,\beta_1,\beta_2},\mathrm{F}^{\beta_3}_{\beta_3,\beta_1,\beta_2},\mathrm{F}^{\beta_2}_{\beta_3,\beta_1,\beta_3},\mathrm{F}^{\beta_4}_{\beta_3,\beta_1,\beta_3},\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_1,\beta_4},\mathrm{F}^{\epsilon}_{\beta_3,\beta_1,\beta_4},$$
$$\mathrm{F}^{\beta_1}_{\beta_3,\beta_1,\beta_4},\mathrm{F}^{\beta_3}_{\beta_3,\beta_1,\beta_4},\mathrm{F}^{\beta_4}_{\beta_3,\beta_1,\beta_4},\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\mathrm{I}},\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\mathrm{I}},\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\epsilon},\mathrm{F}^{\mathrm{I}}_{\beta_3,\alpha_+,\alpha_+},\mathrm{F}^{\epsilon}_{\beta_3,\alpha_+,\alpha_+},\mathrm{F}^{\mathrm{I}}_{\beta_3,\alpha_+,\alpha_-},$$
$$\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\mathrm{I}},\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\mathrm{I}},\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\epsilon},\mathrm{F}^{\mathrm{I}}_{\beta_3,\alpha_-,\alpha_+},\mathrm{F}^{\epsilon}_{\beta_3,\alpha_-,\alpha_+},\mathrm{F}^{\epsilon}_{\beta_3,\alpha_-,\alpha_-},\mathrm{F}^{\beta_1}_{\beta_3,\beta_2,\mathrm{I}},\mathrm{F}^{\beta_4}_{\beta_3,\beta_2,\mathrm{I}},\mathrm{F}^{\beta_4}_{\beta_3,\beta_2,\epsilon},$$
$$\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_2,\beta_1},\mathrm{F}^{\beta_1}_{\beta_3,\beta_2,\beta_1},\mathrm{F}^{\beta_2}_{\beta_3,\beta_2,\beta_1},\mathrm{F}^{\beta_3}_{\beta_3,\beta_2,\beta_1},\mathrm{F}^{\beta_1}_{\beta_3,\beta_2,\beta_2},\mathrm{F}^{\beta_4}_{\beta_3,\beta_2,\beta_2},\mathrm{F}^{\beta_1}_{\beta_3,\beta_2,\beta_3},\mathrm{F}^{\beta_4}_{\beta_3,\beta_2,\beta_3},\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_2,\beta_4},$$
$$\mathrm{F}^{\beta_2}_{\beta_3,\beta_2,\beta_4},\mathrm{F}^{\beta_3}_{\beta_3,\beta_2,\beta_4},\mathrm{F}^{\beta_4}_{\beta_3,\beta_2,\beta_4},\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_3,\mathrm{I}},\mathrm{F}^{\epsilon}_{\beta_3,\beta_3,\mathrm{I}},\mathrm{F}^{\beta_3}_{\beta_3,\beta_3,\mathrm{I}},\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_3,\epsilon},\mathrm{F}^{\epsilon}_{\beta_3,\beta_3,\epsilon},\mathrm{F}^{\beta_2}_{\beta_3,\beta_3,\beta_1},\mathrm{F}^{\beta_4}_{\beta_3,\beta_3,\beta_1},$$
$$\mathrm{F}^{\beta_1}_{\beta_3,\beta_3,\beta_2},\ \mathrm{F}^{\beta_4}_{\beta_3,\beta_3,\beta_2},\ \mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_3,\beta_3},\ \mathrm{F}^{\beta_1}_{\beta_3,\beta_3,\beta_4},\ \mathrm{F}^{\beta_2}_{\beta_3,\beta_3,\beta_4},\ \mathrm{F}^{\beta_1}_{\beta_3,\beta_4,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\beta_3,\beta_4,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\beta_3,\beta_4,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_4,\beta_1},$$
$$\mathrm{F}^{\epsilon}_{\beta_3,\beta_4,\beta_1},\mathrm{F}^{\beta_1}_{\beta_3,\beta_4,\beta_1},\mathrm{F}^{\beta_3}_{\beta_3,\beta_4,\beta_1},\mathrm{F}^{\beta_4}_{\beta_3,\beta_4,\beta_1},\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_4,\beta_2},\mathrm{F}^{\epsilon}_{\beta_3,\beta_4,\beta_2},\mathrm{F}^{\beta_2}_{\beta_3,\beta_4,\beta_2},\mathrm{F}^{\beta_3}_{\beta_3,\beta_4,\beta_2},\mathrm{F}^{\beta_4}_{\beta_3,\beta_4,\beta_2},$$
$$\mathrm{F}^{\beta_1}_{\beta_3,\beta_4,\beta_3},\mathrm{F}^{\beta_2}_{\beta_3,\beta_4,\beta_3},\mathrm{F}^{\beta_1}_{\beta_3,\beta_4,\beta_4},\mathrm{F}^{\beta_2}_{\beta_3,\beta_4,\beta_4},\mathrm{F}^{\beta_4}_{\beta_4,\mathrm{I},\mathrm{I}},\mathrm{F}^{\beta_4}_{\beta_4,\mathrm{I},\epsilon},\mathrm{F}^{\beta_2}_{\beta_4,\mathrm{I},\beta_1},\mathrm{F}^{\beta_3}_{\beta_4,\mathrm{I},\beta_1},\mathrm{F}^{\alpha_+}_{\beta_4,\mathrm{I},\alpha_+},\mathrm{F}^{\alpha_-}_{\beta_4,\mathrm{I},\alpha_+},$$
$$\mathrm{F}^{\alpha_+}_{\beta_4,\mathrm{I},\alpha_-},\ \mathrm{F}^{\alpha_-}_{\beta_4,\mathrm{I},\alpha_-},\ \mathrm{F}^{\beta_1}_{\beta_4,\mathrm{I},\beta_2},\ \mathrm{F}^{\beta_3}_{\beta_4,\mathrm{I},\beta_2},\ \mathrm{F}^{\beta_1}_{\beta_4,\mathrm{I},\beta_3},\ \mathrm{F}^{\beta_2}_{\beta_4,\mathrm{I},\beta_3},\ \mathrm{F}^{\mathrm{I}}_{\beta_4,\mathrm{I},\beta_4},\ \mathrm{F}^{\epsilon}_{\beta_4,\mathrm{I},\beta_4},\ \mathrm{F}^{\beta_4}_{\beta_4,\mathrm{I},\beta_4},\ \mathrm{F}^{\beta_4}_{\beta_4,\epsilon,\mathrm{I}},$$
$$\mathrm{F}^{\beta_4}_{\beta_4,\epsilon,\epsilon},\ \mathrm{F}^{\beta_2}_{\beta_4,\epsilon,\beta_1},\ \mathrm{F}^{\alpha_+}_{\beta_4,\epsilon,\alpha_+},\ \mathrm{F}^{\alpha_-}_{\beta_4,\epsilon,\alpha_+},\ \mathrm{F}^{\alpha_+}_{\beta_4,\epsilon,\alpha_-},\ \mathrm{F}^{\alpha_-}_{\beta_4,\epsilon,\alpha_-},\ \mathrm{F}^{\beta_3}_{\beta_4,\epsilon,\beta_2},\ \mathrm{F}^{\beta_1}_{\beta_4,\epsilon,\beta_3},\ \mathrm{F}^{\mathrm{I}}_{\beta_4,\epsilon,\beta_4},\ \mathrm{F}^{\epsilon}_{\beta_4,\epsilon,\beta_4},$$
$$\mathrm{F}^{\beta_2}_{\beta_4,\beta_1,\mathrm{I}},\ \mathrm{F}^{\beta_3}_{\beta_4,\beta_1,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\beta_4,\beta_1,\epsilon},\ \mathrm{F}^{\beta_3}_{\beta_4,\beta_1,\epsilon},\ \mathrm{F}^{\beta_2}_{\beta_4,\beta_1,\beta_1},\ \mathrm{F}^{\beta_3}_{\beta_4,\beta_1,\beta_1},\ \mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_1,\beta_2},\ \mathrm{F}^{\beta_1}_{\beta_4,\beta_1,\beta_2},\ \mathrm{F}^{\beta_2}_{\beta_4,\beta_1,\beta_2},$$
$$\mathrm{F}^{\beta_4}_{\beta_4,\beta_1,\beta_2},\ \mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_1,\beta_3},\ \mathrm{F}^{\epsilon}_{\beta_4,\beta_1,\beta_3},\ \mathrm{F}^{\beta_1}_{\beta_4,\beta_1,\beta_3},\ \mathrm{F}^{\beta_3}_{\beta_4,\beta_1,\beta_3},\ \mathrm{F}^{\beta_4}_{\beta_4,\beta_1,\beta_3},\ \mathrm{F}^{\beta_2}_{\beta_4,\beta_1,\beta_4},\ \mathrm{F}^{\beta_3}_{\beta_4,\beta_1,\beta_4},\ \mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\mathrm{I}},$$
$$\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\mathrm{I}},\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\epsilon},\mathrm{F}^{\mathrm{I}}_{\beta_4,\alpha_+,\alpha_+},\mathrm{F}^{\epsilon}_{\beta_4,\alpha_+,\alpha_+},\mathrm{F}^{\mathrm{I}}_{\beta_4,\alpha_+,\alpha_-},\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\mathrm{I}},\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\mathrm{I}},\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\epsilon},\mathrm{F}^{\mathrm{I}}_{\beta_4,\alpha_-,\alpha_+},$$
$$\mathrm{F}^{\epsilon}_{\beta_4,\alpha_-,\alpha_+},\ \mathrm{F}^{\epsilon}_{\beta_4,\alpha_-,\alpha_-},\ \mathrm{F}^{\beta_1}_{\beta_4,\beta_2,\mathrm{I}},\ \mathrm{F}^{\beta_3}_{\beta_4,\beta_2,\mathrm{I}},\ \mathrm{F}^{\beta_1}_{\beta_4,\beta_2,\epsilon},\ \mathrm{F}^{\beta_3}_{\beta_4,\beta_2,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_2,\beta_1},\ \mathrm{F}^{\epsilon}_{\beta_4,\beta_2,\beta_1},\ \mathrm{F}^{\beta_1}_{\beta_4,\beta_2,\beta_1},$$
$$\mathrm{F}^{\beta_2}_{\beta_4,\beta_2,\beta_1},\ \mathrm{F}^{\beta_4}_{\beta_4,\beta_2,\beta_1},\ \mathrm{F}^{\beta_1}_{\beta_4,\beta_2,\beta_2},\ \mathrm{F}^{\beta_3}_{\beta_4,\beta_2,\beta_2},\ \mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_2,\beta_3},\ \mathrm{F}^{\beta_2}_{\beta_4,\beta_2,\beta_3},\ \mathrm{F}^{\beta_3}_{\beta_4,\beta_2,\beta_3},\ \mathrm{F}^{\beta_4}_{\beta_4,\beta_2,\beta_3},\ \mathrm{F}^{\beta_1}_{\beta_4,\beta_2,\beta_4},$$
$$\mathrm{F}^{\beta_3}_{\beta_4,\beta_2,\beta_4},\ \mathrm{F}^{\beta_1}_{\beta_4,\beta_3,\mathrm{I}},\ \mathrm{F}^{\beta_2}_{\beta_4,\beta_3,\mathrm{I}},\ \mathrm{F}^{\beta_1}_{\beta_4,\beta_3,\epsilon},\ \mathrm{F}^{\beta_2}_{\beta_4,\beta_3,\epsilon},\ \mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_3,\beta_1},\ \mathrm{F}^{\beta_1}_{\beta_4,\beta_3,\beta_1},\ \mathrm{F}^{\beta_3}_{\beta_4,\beta_3,\beta_1},\ \mathrm{F}^{\beta_4}_{\beta_4,\beta_3,\beta_1},$$
$$\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_3,\beta_2},\mathrm{F}^{\epsilon}_{\beta_4,\beta_3,\beta_2},\mathrm{F}^{\beta_2}_{\beta_4,\beta_3,\beta_2},\mathrm{F}^{\beta_3}_{\beta_4,\beta_3,\beta_2},\mathrm{F}^{\beta_4}_{\beta_4,\beta_3,\beta_2},\mathrm{F}^{\beta_1}_{\beta_4,\beta_3,\beta_3},\mathrm{F}^{\beta_2}_{\beta_4,\beta_3,\beta_3},\mathrm{F}^{\beta_1}_{\beta_4,\beta_3,\beta_4},\mathrm{F}^{\beta_2}_{\beta_4,\beta_3,\beta_4},$$
$$\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_4,\mathrm{I}},\mathrm{F}^{\epsilon}_{\beta_4,\beta_4,\mathrm{I}},\mathrm{F}^{\beta_4}_{\beta_4,\beta_4,\mathrm{I}},\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_4,\epsilon},\mathrm{F}^{\epsilon}_{\beta_4,\beta_4,\epsilon},\mathrm{F}^{\beta_2}_{\beta_4,\beta_4,\beta_1},\mathrm{F}^{\beta_3}_{\beta_4,\beta_4,\beta_1},\mathrm{F}^{\beta_1}_{\beta_4,\beta_4,\beta_2},\mathrm{F}^{\beta_3}_{\beta_4,\beta_4,\beta_2},\mathrm{F}^{\beta_1}_{\beta_4,\beta_4,\beta_3},$$
$$\mathrm{F}^{\beta_2}_{\beta_4,\beta_4,\beta_3},\ \mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_4,\beta_4}$$

A.4.2 Pivotal Structures

There are 2 pivotal structures.

- $\epsilon_{\mathrm{I}} = 1$, $\epsilon_{\epsilon} = 1$, $\epsilon_{\beta_1} = 1$, $\epsilon_{\alpha_+} = 1$, $\epsilon_{\alpha_-} = 1$, $\epsilon_{\beta_2} = 1$, $\epsilon_{\beta_3} = 1$, and $\epsilon_{\beta_4} = 1$ which is spherical with quantum dimensions $q_{\mathrm{I}} = 1$, $q_{\epsilon} = 1$, $q_{\beta_1} = 2$, $q_{\alpha_+} = -3$, $q_{\alpha_-} = -3$, $q_{\beta_2} = 2$, $q_{\beta_3} = 2$, and $q_{\beta_4} = 2$.

- $\epsilon_{\mathrm{I}} = 1$, $\epsilon_{\epsilon} = 1$, $\epsilon_{\beta_1} = 1$, $\epsilon_{\alpha_+} = -1$, $\epsilon_{\alpha_-} = -1$, $\epsilon_{\beta_2} = 1$, $\epsilon_{\beta_3} = 1$, and $\epsilon_{\beta_4} = 1$ which is spherical with quantum dimensions $q_{\mathrm{I}} = 1$, $q_{\epsilon} = 1$, $q_{\beta_1} = 2$, $q_{\alpha_+} = 3$, $q_{\alpha_-} = 3$, $q_{\beta_2} = 2$, $q_{\beta_3} = 2$, and $q_{\beta_4} = 2$.

A.4.3 Braidings

There is 1 braidings which corresponds to this solution.

- $\mathrm{R}^{\alpha_+}_{\alpha_-,\beta_3} = \mathrm{R}^{\alpha_+}_{\alpha_-,\beta_4} = \mathrm{R}^{\alpha_+}_{\beta_3,\alpha_-} = \mathrm{R}^{\alpha_+}_{\beta_4,\alpha_-} = \begin{pmatrix} -\sqrt[6]{-1} \end{pmatrix}$
- $R^{c}_{a,b} = \begin{pmatrix} \sqrt[6]{-1} \end{pmatrix}$
  - $\mathrm{R}^{\beta_1}_{\alpha_+,\alpha_-}$, $\mathrm{R}^{\beta_2}_{\alpha_+,\alpha_-}$, $\mathrm{R}^{\alpha_-}_{\alpha_+,\beta_3}$, $\mathrm{R}^{\alpha_-}_{\alpha_+,\beta_4}$, $\mathrm{R}^{\beta_1}_{\alpha_-,\alpha_+}$, $\mathrm{R}^{\beta_2}_{\alpha_-,\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\beta_3,\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\beta_4,\alpha_+}$
- $\mathrm{R}^{\beta_3}_{\alpha_+,\alpha_+} = \mathrm{R}^{\beta_4}_{\alpha_+,\alpha_+} = \mathrm{R}^{\epsilon}_{\beta_3,\beta_3} = \mathrm{R}^{\epsilon}_{\beta_4,\beta_4} = \begin{pmatrix} \frac{1}{2}\left(1 + i\sqrt{3}\right) \end{pmatrix}$
- $\mathrm{R}^{\alpha_+}_{\beta_1,\alpha_-} = \mathrm{R}^{\alpha_+}_{\alpha_-,\beta_1} = \mathrm{R}^{\alpha_+}_{\alpha_-,\beta_2} = \mathrm{R}^{\alpha_+}_{\beta_2,\alpha_-} = \begin{pmatrix} -(-1)^{5/6} \end{pmatrix}$
- $R^{c}_{a,b} = \begin{pmatrix} (-1)^{5/6} \end{pmatrix}$
  - $\mathrm{R}^{\alpha_-}_{\beta_1,\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\alpha_+,\beta_1}$, $\mathrm{R}^{\beta_3}_{\alpha_+,\alpha_-}$, $\mathrm{R}^{\beta_4}_{\alpha_+,\alpha_-}$, $\mathrm{R}^{\alpha_-}_{\alpha_+,\beta_2}$, $\mathrm{R}^{\beta_3}_{\alpha_-,\alpha_+}$, $\mathrm{R}^{\beta_4}_{\alpha_-,\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\beta_2,\alpha_+}$
- $R^{c}_{a,b} = \begin{pmatrix} -\frac{1}{2}i\left(-i + \sqrt{3}\right) \end{pmatrix}$
  - $\mathrm{R}^{\beta_1}_{\beta_1,\beta_1}$, $\mathrm{R}^{\alpha_+}_{\beta_1,\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\beta_1,\alpha_-}$, $\mathrm{R}^{\beta_4}_{\beta_1,\beta_3}$, $\mathrm{R}^{\beta_3}_{\beta_1,\beta_4}$, $\mathrm{R}^{\alpha_+}_{\alpha_+,\beta_1}$, $\mathrm{R}^{\alpha_+}_{\alpha_+,\beta_2}$, $\mathrm{R}^{\alpha_-}_{\alpha_-,\beta_1}$, $\mathrm{R}^{\beta_3}_{\alpha_-,\alpha_-}$, $\mathrm{R}^{\beta_4}_{\alpha_-,\alpha_-}$, $\mathrm{R}^{\alpha_-}_{\alpha_-,\beta_2}$, $\mathrm{R}^{\alpha_+}_{\beta_2,\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\beta_2,\alpha_-}$, $\mathrm{R}^{\beta_2}_{\beta_2,\beta_2}$, $\mathrm{R}^{\beta_4}_{\beta_2,\beta_3}$, $\mathrm{R}^{\beta_3}_{\beta_2,\beta_4}$, $\mathrm{R}^{\beta_4}_{\beta_3,\beta_1}$, $\mathrm{R}^{\beta_4}_{\beta_3,\beta_2}$, $\mathrm{R}^{\mathrm{I}}_{\beta_3,\beta_3}$, $\mathrm{R}^{\beta_3}_{\beta_4,\beta_1}$, $\mathrm{R}^{\beta_3}_{\beta_4,\beta_2}$, $\mathrm{R}^{\mathrm{I}}_{\beta_4,\beta_4}$

- $\mathrm{R}^{\epsilon}_{\beta_1,\beta_1} = \mathrm{R}^{\beta_1}_{\alpha_+,\alpha_+} = \mathrm{R}^{\beta_2}_{\alpha_+,\alpha_+} = \mathrm{R}^{\epsilon}_{\beta_2,\beta_2} = \begin{pmatrix} \frac{1}{2}\left(1 - i\sqrt{3}\right) \end{pmatrix}$

- $R^{c}_{a,b} = \begin{pmatrix} \frac{1}{2}i\left(i + \sqrt{3}\right) \end{pmatrix}$

  - $\mathrm{R}^{\mathrm{I}}_{\beta_1,\beta_1}$, $\mathrm{R}^{\beta_2}_{\beta_1,\beta_3}$, $\mathrm{R}^{\beta_2}_{\beta_1,\beta_4}$, $\mathrm{R}^{\alpha_+}_{\alpha_+,\beta_3}$, $\mathrm{R}^{\alpha_+}_{\alpha_+,\beta_4}$, $\mathrm{R}^{\beta_1}_{\alpha_-,\alpha_-}$, $\mathrm{R}^{\beta_2}_{\alpha_-,\alpha_-}$, $\mathrm{R}^{\alpha_-}_{\alpha_-,\beta_3}$, $\mathrm{R}^{\alpha_-}_{\alpha_-,\beta_4}$, $\mathrm{R}^{\mathrm{I}}_{\beta_2,\beta_2}$, $\mathrm{R}^{\beta_1}_{\beta_2,\beta_3}$, $\mathrm{R}^{\beta_1}_{\beta_2,\beta_4}$, $\mathrm{R}^{\beta_2}_{\beta_3,\beta_1}$, $\mathrm{R}^{\alpha_+}_{\beta_3,\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\beta_3,\alpha_-}$, $\mathrm{R}^{\beta_1}_{\beta_3,\beta_2}$, $\mathrm{R}^{\beta_3}_{\beta_3,\beta_3}$, $\mathrm{R}^{\beta_2}_{\beta_4,\beta_1}$, $\mathrm{R}^{\alpha_+}_{\beta_4,\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\beta_4,\alpha_-}$, $\mathrm{R}^{\beta_1}_{\beta_4,\beta_2}$, $\mathrm{R}^{\beta_4}_{\beta_4,\beta_4}$

- $\mathrm{R}^{\alpha_+}_{\epsilon,\alpha_-} = \mathrm{R}^{\alpha_+}_{\alpha_-,\epsilon} = \begin{pmatrix} -i \end{pmatrix}$

- $\mathrm{R}^{\alpha_-}_{\epsilon,\alpha_+} = \mathrm{R}^{\alpha_-}_{\alpha_+,\epsilon} = \mathrm{R}^{\epsilon}_{\alpha_+,\alpha_-} = \mathrm{R}^{\epsilon}_{\alpha_-,\alpha_+} = \begin{pmatrix} i \end{pmatrix}$

- $R^{c}_{a,b} = \begin{pmatrix} -1 \end{pmatrix}$

  - $\mathrm{R}^{\beta_1}_{\epsilon,\beta_1}$, $\mathrm{R}^{\beta_2}_{\epsilon,\beta_2}$, $\mathrm{R}^{\beta_3}_{\epsilon,\beta_3}$, $\mathrm{R}^{\beta_4}_{\epsilon,\beta_4}$, $\mathrm{R}^{\beta_1}_{\beta_1,\epsilon}$, $\mathrm{R}^{\mathrm{I}}_{\alpha_+,\alpha_+}$, $\mathrm{R}^{\beta_2}_{\beta_2,\epsilon}$, $\mathrm{R}^{\beta_3}_{\beta_3,\epsilon}$, $\mathrm{R}^{\beta_4}_{\beta_4,\epsilon}$

- $R^{c}_{a,b} = \begin{pmatrix} 1 \end{pmatrix}$

  - $\mathrm{R}^{\mathrm{I}}_{\mathrm{I},\mathrm{I}}$, $\mathrm{R}^{\epsilon}_{\mathrm{I},\epsilon}$, $\mathrm{R}^{\beta_1}_{\mathrm{I},\beta_1}$, $\mathrm{R}^{\alpha_+}_{\mathrm{I},\alpha_+}$, $\mathrm{R}^{\alpha_-}_{\mathrm{I},\alpha_-}$, $\mathrm{R}^{\beta_2}_{\mathrm{I},\beta_2}$, $\mathrm{R}^{\beta_3}_{\mathrm{I},\beta_3}$, $\mathrm{R}^{\beta_4}_{\mathrm{I},\beta_4}$, $\mathrm{R}^{\epsilon}_{\epsilon,\mathrm{I}}$, $\mathrm{R}^{\mathrm{I}}_{\epsilon,\epsilon}$, $\mathrm{R}^{\beta_1}_{\beta_1,\mathrm{I}}$, $\mathrm{R}^{\beta_3}_{\beta_1,\beta_2}$, $\mathrm{R}^{\beta_4}_{\beta_1,\beta_2}$, $\mathrm{R}^{\alpha_+}_{\alpha_+,\mathrm{I}}$, $\mathrm{R}^{\alpha_-}_{\alpha_-,\mathrm{I}}$, $\mathrm{R}^{\mathrm{I}}_{\alpha_-,\alpha_-}$, $\mathrm{R}^{\beta_2}_{\beta_2,\mathrm{I}}$, $\mathrm{R}^{\beta_3}_{\beta_2,\beta_1}$, $\mathrm{R}^{\beta_4}_{\beta_2,\beta_1}$, $\mathrm{R}^{\beta_3}_{\beta_3,\mathrm{I}}$, $\mathrm{R}^{\beta_1}_{\beta_3,\beta_4}$, $\mathrm{R}^{\beta_2}_{\beta_3,\beta_4}$, $\mathrm{R}^{\beta_4}_{\beta_4,\mathrm{I}}$, $\mathrm{R}^{\beta_1}_{\beta_4,\beta_3}$, $\mathrm{R}^{\beta_2}_{\beta_4,\beta_3}$

A.4.4 Balanced, Ribbon, and Modular Structures

There are 2 balanced categories which correspond to this solution. Both of them are modular.

Balanced Category 1 has pivotal structure 1. Its T-Matrix is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) \end{pmatrix}.$$

It is ribbon and has S-Matrix given by:

$$\begin{pmatrix} 1 & 1 & 2 & -3 & -3 & 2 & 2 & 2 \\ 1 & 1 & 2 & 3 & 3 & 2 & 2 & 2 \\ 2 & 2 & -2 & 0 & 0 & 4 & -2 & -2 \\ -3 & 3 & 0 & -3 & 3 & 0 & 0 & 0 \\ -3 & 3 & 0 & 3 & -3 & 0 & 0 & 0 \\ 2 & 2 & 4 & 0 & 0 & -2 & -2 & -2 \\ 2 & 2 & -2 & 0 & 0 & -2 & -2 & 4 \\ 2 & 2 & -2 & 0 & 0 & -2 & 4 & -2 \end{pmatrix}.$$

Balanced Category 2 has pivotal structure 2. Its T-Matrix is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}i\left(-i+\sqrt{3}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}i\left(i+\sqrt{3}\right) \end{pmatrix}.$$

It is ribbon and has S-Matrix given by:

$$\begin{pmatrix} 1 & 1 & 2 & 3 & 3 & 2 & 2 & 2 \\ 1 & 1 & 2 & -3 & -3 & 2 & 2 & 2 \\ 2 & 2 & -2 & 0 & 0 & 4 & -2 & -2 \\ 3 & -3 & 0 & -3 & 3 & 0 & 0 & 0 \\ 3 & -3 & 0 & 3 & -3 & 0 & 0 & 0 \\ 2 & 2 & 4 & 0 & 0 & -2 & -2 & -2 \\ 2 & 2 & -2 & 0 & 0 & -2 & -2 & 4 \\ 2 & 2 & -2 & 0 & 0 & -2 & 4 & -2 \end{pmatrix}.$$

A.5 Solution 5 – Hagge-Titsworth Category

A.5.1 F-Matrices

- $\mathrm{F}^{\alpha_-}_{\alpha_-,\alpha_-,\alpha_-} = \begin{pmatrix} -\frac{1}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$

- $\mathrm{F}^{\alpha_+}_{\alpha_-,\alpha_-,\alpha_+} = \begin{pmatrix} \frac{1}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$

- $\mathrm{F}^{\alpha_+}_{\alpha_-,\alpha_+,\alpha_-} = \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

- $\mathrm{F}^{\alpha_-}_{\alpha_-,\alpha_+,\alpha_+} = \begin{pmatrix} -\frac{1}{3} & -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

- $\mathrm{F}^{\alpha_+}_{\alpha_+,\alpha_-,\alpha_-} = \begin{pmatrix} -\frac{1}{3} & \frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & -\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{\sqrt{2}}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$
- $\mathrm{F}^{\alpha_-}_{\alpha_+,\alpha_-,\alpha_+} = \begin{pmatrix} -\frac{1}{3} & \frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & -\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{\sqrt{2}}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$
- $\mathrm{F}^{\alpha_-}_{\alpha_+,\alpha_+,\alpha_-} = \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{\sqrt{2}}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$
- $\mathrm{F}^{\alpha_+}_{\alpha_+,\alpha_+,\alpha_+} = \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{\sqrt{2}}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{\sqrt{2}}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{\sqrt{2}}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$
- $\mathrm{F}^{\alpha_-}_{\alpha_-,\alpha_-,\alpha_+} = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}$

- $\mathrm{F}^{\alpha_+}_{\alpha_-,\alpha_+,\alpha_+} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix}$
- $\mathrm{F}^{\alpha_-}_{\alpha_+,\alpha_-,\alpha_-} = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 \end{pmatrix}$
- $\mathrm{F}^{\alpha_+}_{\alpha_+,\alpha_-,\alpha_+} = \mathrm{F}^{\alpha_-}_{\alpha_-,\alpha_+,\alpha_-} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 \end{pmatrix}$
- $\mathrm{F}^{\alpha_+}_{\alpha_+,\alpha_+,\alpha_-} = \begin{pmatrix} 0 & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 \end{pmatrix}$
- $\mathrm{F}^{\alpha_-}_{\alpha_+,\alpha_+,\alpha_+} = \mathrm{F}^{\alpha_+}_{\alpha_-,\alpha_-,\alpha_-} = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 \end{pmatrix}$
- $\mathrm{F}^{\beta_2}_{\beta_2,\beta_2,\beta_2} = \mathrm{F}^{\beta_3}_{\beta_3,\beta_3,\beta_3} = \mathrm{F}^{\beta_4}_{\beta_4,\beta_4,\beta_4} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$

- $\mathrm{F}^{\beta_1}_{\beta_1,\beta_1,\beta_1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix}$
- $F^d_{abc} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\beta_1}, \mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\beta_4}, \mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\beta_1}, \mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\beta_4}, \mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\beta_1}, \mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\beta_2}, \mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\beta_1}, \mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\beta_2}, \mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\beta_1}, \mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\beta_3}, \mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\beta_1}, \mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\beta_3}$
- $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_2,\alpha_+} = \mathrm{F}^{\beta_2}_{\alpha_-,\beta_3,\alpha_+} = \mathrm{F}^{\beta_3}_{\alpha_-,\beta_4,\alpha_+} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$
- $F^d_{abc} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
  - $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_1,\alpha_-}, \mathrm{F}^{\beta_3}_{\alpha_-,\beta_1,\alpha_-}, \mathrm{F}^{\beta_4}_{\alpha_-,\beta_1,\alpha_-}, \mathrm{F}^{\beta_1}_{\alpha_-,\beta_2,\alpha_-}, \mathrm{F}^{\beta_3}_{\alpha_-,\beta_2,\alpha_-}, \mathrm{F}^{\beta_1}_{\alpha_-,\beta_3,\alpha_-}, \mathrm{F}^{\beta_4}_{\alpha_-,\beta_3,\alpha_-}, \mathrm{F}^{\beta_1}_{\alpha_-,\beta_4,\alpha_-}, \mathrm{F}^{\beta_2}_{\alpha_-,\beta_4,\alpha_-}, \mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\beta_1}, \mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\beta_4}, \mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\beta_1}, \mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\beta_2}, \mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\beta_1}, \mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\beta_3}$
- $F^d_{abc} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$
  - $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_1,\alpha_+}, \mathrm{F}^{\beta_3}_{\alpha_-,\beta_1,\alpha_+}, \mathrm{F}^{\beta_4}_{\alpha_-,\beta_1,\alpha_+}, \mathrm{F}^{\beta_1}_{\alpha_-,\beta_2,\alpha_+}, \mathrm{F}^{\beta_3}_{\alpha_-,\beta_2,\alpha_+}, \mathrm{F}^{\beta_1}_{\alpha_-,\beta_3,\alpha_+}, \mathrm{F}^{\beta_4}_{\alpha_-,\beta_3,\alpha_+}, \mathrm{F}^{\beta_1}_{\alpha_-,\beta_4,\alpha_+}, \mathrm{F}^{\beta_2}_{\alpha_-,\beta_4,\alpha_+}$
- $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_1,\alpha_+} = \mathrm{F}^{\beta_2}_{\alpha_-,\beta_2,\alpha_+} = \mathrm{F}^{\beta_3}_{\alpha_-,\beta_3,\alpha_+} = \mathrm{F}^{\beta_4}_{\alpha_-,\beta_4,\alpha_+} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_2,\alpha_-} = \mathrm{F}^{\beta_2}_{\alpha_+,\beta_3,\alpha_-} = \mathrm{F}^{\beta_3}_{\alpha_+,\beta_4,\alpha_-} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

- $F^d_{abc} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$
  - $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\beta_3}$
- $F^d_{abc} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}$
  - $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_4,\alpha_-}$
- $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_1,\alpha_-} = \mathrm{F}^{\beta_2}_{\alpha_+,\beta_2,\alpha_-} = \mathrm{F}^{\beta_3}_{\alpha_+,\beta_3,\alpha_-} = \mathrm{F}^{\beta_4}_{\alpha_+,\beta_4,\alpha_-} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
- $F^d_{abc} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$
  - $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\beta_2}$
- $F^d_{abc} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$
  - $\mathrm{F}^{\beta_1}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_4,\alpha_-,\alpha_-}$
- $F^d_{abc} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

- $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\beta_4}$

- $F^d_{abc} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

  - $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\beta_2}$

- $F^d_{abc} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

  - $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\beta_2}$

- $F^d_{abc} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

  - $\mathrm{F}^{\beta_1}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_4,\alpha_-,\alpha_+}$

- $F^d_{abc} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

  - $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_2}_{\beta_1,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_1,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_1,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_2,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_2,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_1,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_3,\beta_4}$

- $F^d_{abc} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

  - $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\beta_4}$

- $F^d_{abc} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

  - $\mathrm{F}^{\beta_3}_{\beta_1,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_1}_{\beta_2,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_2,\beta_2,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_2,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_2,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_1,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_3,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_4,\beta_1}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_4,\beta_3}$

- $F^d_{abc} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

  - $\mathrm{F}^{\beta_2}_{\beta_1,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_1,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_1,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_3,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_1,\beta_1}$

- $F^d_{abc} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

  - $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_1,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_1,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_4,\alpha_-}$

- $F^d_{abc} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

  - $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_4,\alpha_+}$

- $F^d_{abc} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$

  - $\mathrm{F}^{\alpha_-}_{\beta_1,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_4,\alpha_-}$

- $F^d_{abc} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

  - $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_1,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\beta_3}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\beta_2}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_2}_{\alpha_+,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_+,\alpha_-,\beta_4}$, $\mathrm{F}^{\beta_4}_{\alpha_+,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_2,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_2,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_2,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\beta_3,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_1,\beta_1}$, $\mathrm{F}^{\beta_2}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\beta_2,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\beta_4,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_4,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_1,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_2,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_2,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_3,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\beta_4,\beta_4,\alpha_+}$

- $F^d_{abc} = \begin{pmatrix} -1 \end{pmatrix}$

– $\mathrm{F}^{\beta_1}_{\epsilon,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_1,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_1,\beta_2}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_1,\beta_3}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\beta_1}$,
$\mathrm{F}^{\epsilon}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\epsilon,\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\epsilon,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\epsilon}$,
$\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\epsilon,\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\epsilon,\alpha_-,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\alpha_-,\beta_4}$,
$\mathrm{F}^{\alpha_+}_{\epsilon,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_2,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_2,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_2,\alpha_-}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\beta_2,\beta_2}$, $\mathrm{F}^{\epsilon}_{\epsilon,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_2,\beta_2}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_2,\beta_4}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_2,\beta_4}$,
$\mathrm{F}^{\alpha_+}_{\epsilon,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_3,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_3,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_3,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_3,\beta_2}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_3,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\beta_3,\beta_3}$, $\mathrm{F}^{\epsilon}_{\epsilon,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_3,\beta_4}$,
$\mathrm{F}^{\alpha_+}_{\epsilon,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_4,\alpha_+}$, $\mathrm{F}^{\alpha_+}_{\epsilon,\beta_4,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\epsilon,\beta_4,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\epsilon,\beta_4,\beta_2}$, $\mathrm{F}^{\beta_1}_{\epsilon,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_2}_{\epsilon,\beta_4,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\epsilon,\beta_4,\beta_4}$, $\mathrm{F}^{\epsilon}_{\epsilon,\beta_4,\beta_4}$, $\mathrm{F}^{\beta_4}_{\epsilon,\beta_4,\beta_4}$,
$\mathrm{F}^{\beta_1}_{\beta_1,\epsilon,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_1,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_1,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_3}_{\beta_1,\epsilon,\beta_4}$, $\mathrm{F}^{\beta_1}_{\beta_1,\beta_1,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\beta_1,\alpha_+,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_1,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_1,\alpha_-,\epsilon}$,
$\mathrm{F}^{\mathrm{I}}_{\beta_1,\alpha_-,\alpha_-}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_2,\beta_4}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_3,\beta_2}$, $\mathrm{F}^{\epsilon}_{\beta_1,\beta_4,\beta_3}$, $\mathrm{F}^{\epsilon}_{\alpha_+,\epsilon,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\alpha_+,\epsilon,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_1,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_1,\epsilon}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_+,\epsilon}$,
$\mathrm{F}^{\epsilon}_{\alpha_+,\alpha_+,\beta_1}$, $\mathrm{F}^{\beta_1}_{\alpha_+,\alpha_-,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\alpha_+,\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_2,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_2,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_3,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_3,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\alpha_+,\beta_4,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_+,\beta_4,\epsilon}$,
$\mathrm{F}^{\mathrm{I}}_{\alpha_-,\epsilon,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\alpha_-,\epsilon,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_1,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_1,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_+,\epsilon}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_+,\epsilon}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_+,\epsilon}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_+,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_+,\beta_1}$,
$\mathrm{F}^{\epsilon}_{\alpha_-,\alpha_+,\beta_1}$, $\mathrm{F}^{\epsilon}_{\alpha_-,\alpha_+,\beta_2}$, $\mathrm{F}^{\epsilon}_{\alpha_-,\alpha_+,\beta_3}$, $\mathrm{F}^{\epsilon}_{\alpha_-,\alpha_+,\beta_4}$, $\mathrm{F}^{\epsilon}_{\alpha_-,\alpha_-,\epsilon}$, $\mathrm{F}^{\beta_2}_{\alpha_-,\alpha_-,\epsilon}$, $\mathrm{F}^{\beta_3}_{\alpha_-,\alpha_-,\epsilon}$, $\mathrm{F}^{\beta_4}_{\alpha_-,\alpha_-,\epsilon}$,
$\mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\epsilon}_{\alpha_-,\alpha_-,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_-,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_-,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\alpha_-,\alpha_-,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_2,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_2,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_3,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_3,\epsilon}$,
$\mathrm{F}^{\alpha_+}_{\alpha_-,\beta_4,\epsilon}$, $\mathrm{F}^{\alpha_-}_{\alpha_-,\beta_4,\epsilon}$, $\mathrm{F}^{\beta_4}_{\beta_2,\epsilon,\beta_1}$, $\mathrm{F}^{\beta_2}_{\beta_2,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_4}_{\beta_2,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_3}_{\beta_2,\epsilon,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_2,\beta_1,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\beta_2,\alpha_+,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_2,\alpha_+,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\beta_2,\alpha_+,\alpha_-}$,
$\mathrm{F}^{\epsilon}_{\beta_2,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_2,\alpha_-,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_2,\alpha_-,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\epsilon}_{\beta_2,\alpha_-,\alpha_-}$, $\mathrm{F}^{\mathrm{I}}_{\beta_2,\beta_2,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_2,\epsilon}$, $\mathrm{F}^{\beta_2}_{\beta_2,\beta_2,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_2,\beta_2}$,
$\mathrm{F}^{\beta_1}_{\beta_2,\beta_3,\epsilon}$, $\mathrm{F}^{\beta_4}_{\beta_2,\beta_3,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_3,\beta_1}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_3,\beta_4}$, $\mathrm{F}^{\beta_3}_{\beta_2,\beta_4,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_2,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_3,\epsilon,\beta_1}$, $\mathrm{F}^{\beta_4}_{\beta_3,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_3}_{\beta_3,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_3,\epsilon,\beta_4}$,
$\mathrm{F}^{\beta_4}_{\beta_3,\beta_1,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\beta_3,\alpha_+,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_3,\alpha_+,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\epsilon}_{\beta_3,\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\beta_3,\alpha_-,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\alpha_-,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\alpha_-,\alpha_-}$, $\mathrm{F}^{\epsilon}_{\beta_3,\alpha_-,\alpha_-}$,
$\mathrm{F}^{\beta_4}_{\beta_3,\beta_2,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_2,\beta_4}$, $\mathrm{F}^{\mathrm{I}}_{\beta_3,\beta_3,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_3,\epsilon}$, $\mathrm{F}^{\beta_3}_{\beta_3,\beta_3,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_3,\beta_3}$, $\mathrm{F}^{\beta_1}_{\beta_3,\beta_4,\epsilon}$, $\mathrm{F}^{\beta_2}_{\beta_3,\beta_4,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_4,\beta_1}$, $\mathrm{F}^{\epsilon}_{\beta_3,\beta_4,\beta_2}$,
$\mathrm{F}^{\beta_3}_{\beta_4,\epsilon,\beta_1}$, $\mathrm{F}^{\beta_3}_{\beta_4,\epsilon,\beta_2}$, $\mathrm{F}^{\beta_2}_{\beta_4,\epsilon,\beta_3}$, $\mathrm{F}^{\beta_4}_{\beta_4,\epsilon,\beta_4}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_1,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\beta_4,\alpha_+,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_4,\alpha_+,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\alpha_+,\alpha_-}$, $\mathrm{F}^{\epsilon}_{\beta_4,\alpha_+,\alpha_-}$,
$\mathrm{F}^{\alpha_-}_{\beta_4,\alpha_-,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\alpha_-,\alpha_+}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\epsilon}_{\beta_4,\alpha_-,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\beta_4,\beta_2,\epsilon}$, $\mathrm{F}^{\beta_3}_{\beta_4,\beta_2,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_2,\beta_1}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_2,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_3,\epsilon}$,
$\mathrm{F}^{\epsilon}_{\beta_4,\beta_3,\beta_2}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_4,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_4,\epsilon}$, $\mathrm{F}^{\beta_4}_{\beta_4,\beta_4,\epsilon}$, $\mathrm{F}^{\epsilon}_{\beta_4,\beta_4,\beta_4}$

- $F^d_{abc} = \begin{pmatrix} 1 \end{pmatrix}$

– $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\mathrm{I},\mathrm{I}}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\mathrm{I},\epsilon}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\mathrm{I},\beta_1}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\mathrm{I},\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\mathrm{I},\alpha_-}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\mathrm{I},\beta_2}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\mathrm{I},\beta_3}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\mathrm{I},\beta_4}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\epsilon,\mathrm{I}}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\epsilon,\epsilon}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\epsilon,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\epsilon,\alpha_+}$,
$\mathrm{F}^{\alpha_+}_{\mathrm{I},\epsilon,\alpha_-}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\epsilon,\beta_2}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\epsilon,\beta_3}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\epsilon,\beta_4}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_1,\mathrm{I}}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_1,\epsilon}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\beta_1,\beta_1}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\beta_1,\beta_1}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\beta_1,\beta_1}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_1,\alpha_+}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_1,\alpha_+}$,
$\mathrm{F}^{\alpha_+}_{\mathrm{I},\beta_1,\alpha_-}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\beta_1,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_1,\beta_2}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_1,\beta_2}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_1,\beta_3}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\beta_1,\beta_3}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\beta_1,\beta_4}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\beta_1,\beta_4}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\mathrm{I}}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\epsilon}$,
$\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\beta_1}$, $\mathrm{F}^{\mathrm{I}}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\alpha_+,\alpha_+}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\alpha_+,\alpha_-}$,

$\mathrm{F}^{\beta_2}_{\mathrm{I},\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\alpha_+,\alpha_-}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\alpha_+,\alpha_-}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\beta_2}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\beta_3}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_+,\beta_4}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_+,\beta_4}$,
$\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_-,\mathrm{I}}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_-,\epsilon}$, $\mathrm{F}^{\alpha_+}_{\mathrm{I},\alpha_-,\beta_1}$, $\mathrm{F}^{\alpha_-}_{\mathrm{I},\alpha_-,\beta_1}$, $\mathrm{F}^{\epsilon}_{\mathrm{I},\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_1}_{\mathrm{I},\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_2}_{\mathrm{I},\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_3}_{\mathrm{I},\alpha_-,\alpha_+}$, $\mathrm{F}^{\beta_4}_{\mathrm{I},\alpha_-,\alpha_+}$,
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$\mathrm{F}^{\beta_1}_{\beta_4,\beta_4,\beta_3}$, $\mathrm{F}^{\beta_2}_{\beta_4,\beta_4,\beta_3}$, $\mathrm{F}^{\mathrm{I}}_{\beta_4,\beta_4,\beta_4}$

A.5.2 Pivotal Structures

There are 2 pivotal structures.

- $\epsilon_{\mathrm{I}} = 1$, $\epsilon_{\epsilon} = 1$, $\epsilon_{\beta_1} = 1$, $\epsilon_{\alpha_+} = 1$, $\epsilon_{\alpha_-} = -1$, $\epsilon_{\beta_2} = 1$, $\epsilon_{\beta_3} = 1$, and $\epsilon_{\beta_4} = 1$ which is spherical with quantum dimensions $q_{\mathrm{I}} = 1$, $q_{\epsilon} = 1$, $q_{\beta_1} = 2$, $q_{\alpha_+} = 3$, $q_{\alpha_-} = 3$, $q_{\beta_2} = 2$, $q_{\beta_3} = 2$, and $q_{\beta_4} = 2$.

- $\epsilon_{\mathrm{I}} = 1$, $\epsilon_{\epsilon} = 1$, $\epsilon_{\beta_1} = 1$, $\epsilon_{\alpha_+} = -1$, $\epsilon_{\alpha_-} = 1$, $\epsilon_{\beta_2} = 1$, $\epsilon_{\beta_3} = 1$, and $\epsilon_{\beta_4} = 1$ which is spherical with quantum dimensions $q_{\mathrm{I}} = 1$, $q_{\epsilon} = 1$, $q_{\beta_1} = 2$, $q_{\alpha_+} = -3$, $q_{\alpha_-} = -3$, $q_{\beta_2} = 2$, $q_{\beta_3} = 2$, and $q_{\beta_4} = 2$.

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