1 Introduction

I work at the intersection of representation theory, quantum algebra, and mathematical physics, essentially using mathematics descended from the Jones polynomial [Jon85] to describe physics descended from Dirac's magnetic monopole [Dir31]. My focus is on the classification of fusion categories [ENO05]. These capture algebraic properties of anyons, exotic quasi-particles realised in quantum phases of matter.

For various reasons, fusion categories are usually studied from algebraic and topological view points. However, all of their structure is encapsulated by arithmetic data i.e. solutions to polynomial equations. These facts suggest geometric and arithmetic approaches to classification of fusion categories which is what I am pursuing. This has born fruit: fusion categories can be classified by geometric invariants [HT15].

Physicists make use of fusion categories concretely via solutions. Levin-Wen [LW05] and Kitaev-Kong [KK12] models realizing (extended) 2+1 TQFTs take arithmetic data for a fusion category as input. This data is also used to construct braid group representations for topological quantum computing [Wan10].

The polynomial systems involved can be very large - 10^3 variables in 10^5 equations is typical. This is beyond the ken of general purpose Gröbner basis algorithms which must be augmented with specialized techniques. To this end, I've helped develop the FusionCategories [HT] Mathematica package for solving these equations and manipulating this data. This has led to collaborations with both physicists and mathematicians to push forward on classification of fusion categories.

2 Background

Fusion categories over a field k generalize categories of finite group representations; they're tensor categories with finitely many classes of simple objects, all having duals. They arise in representation theory [Kas95], operator algebras [EK98], topological quantum field theories [BK01], and quantum invariants of 3-manifolds [Tur10].

The tensor structure of a fusion category $\mathcal{C}$ give rise to the Grothendieck ring $\mathcal{K}_0(\mathcal{C})$. Ocneanu Rigidity [ENO05] guarantees that there are only finitely many fusion categories for any given Grothendieck ring. It is conjectured [Ost03b] that there are only finitely many Grothendieck rings for a given number of simple objects.

Given a fusion category $\mathcal{C}$ over k, there exist triples (B, N, F) [DHW13] - known as arithmetic data for $\mathcal{C}$ - from which it can be completely reconstructed. B is a set of labels for isomorphism classes of objects in $\mathcal{C}$. N is a set of non-negative integers encoding the structure constants of $\mathcal{K}_0(\mathcal{C})$. These are solutions to diophantine equations. F is a set of numbers which encodes the associator information. These are solutions to polynomial equations, called pentagon equations, taking values in k.

3 Classification by Geometric Invariants

The numbers F define points in an algebraic scheme X acted upon by a reductive group $\mathcal{G}$: a semi-direct product of a finite group G with a direct product of general linear groups. Fusion categories are in $1-1$ correspondence with orbits of $\mathcal{G}$ in X, which are irreducible. This leads to the following:

Theorem 3.1 ([HT15, Theorems 1.1 and 1.5]). *If k has characteristic 0, then $X/\mathcal{G}$ is a geometric quotient and there always exist $\mathcal{G}$-invariant regular functions [MFK94] strong enough to classify arbitrary fusion categories.*

Geometrically, $X/\mathcal{G}$ is a moduli space whose points are essentially fusion categories. Algebraically, if $\mathcal{O}(X)$ is the algebra of global sections, then localizations of the invariant sub-algebra $\mathcal{O}(X)^{\mathcal{G}}$ at maximal ideals are complete invariants of the corresponding fusion categories. The $\mathcal{O}(X)^{\mathcal{G}}$ contain a wealth of information: categorical invariants that can be written using arithmetic data are geometric invariants and live in $\mathcal{O}(X)^{\mathcal{G}}$.

If the categories are multiplicity-free (i.e. all non-zero $hom(a \otimes b, c)$ spaces are one dimensional), we show that the problem can be partitioned so that there is a fast algorithm for computing invariants. E. Ardonne and P. Finch and I [AFT] use this to construct invariants which classify, for all $p \geq 1$, a two parameter family of categories Grothendieck equivalent to $B_{p,2}$, the representations of the (untwisted) affine Kac-Moody algebra $B_p^{(1)}$ with highest integral weight 2. The number of categories yields an integer sequence not in the On-Line Encyclopedia of Integer Sequences [OEI].

3.1 Future Work

My long term goal is to classify and develop tools for classifying fusion categories. As a start, I think the above work indicates that considering the geometric objects associated to fusion categories will yield new information. The category $\mathcal{C}$ corresponds to a set of irreducible components $X(\mathcal{C})$ of X which contains all of the information necessary to construct a category equivalent to $\mathcal{C}$. If $\mathcal{C}$ is taken over $\mathbb{C}$, the geometric points of $X(\mathcal{C})$ form a complex manifold $X'(\mathcal{C})$ and tools for studying these are well developed. There is a transitive action of G on $X'(\mathcal{C})$ and so one could also look at $X'(\mathcal{C})/G$ which, since G does not always act freely, may not be a manifold.

More immediately there are tasks open in applying invariant theory to fusion categories. One is the task of finding a construction for invariants in cases with multiplicity. Another is proving the existence of classifying invariants in positive characteristic. Something more speculative would be to develop an algorithm for computing $\mathcal{O}(X)^{\mathcal{G}}$ and its localizations without first solving pentagon equations. We computed this completely for the case of Fibonacci categories, however there are hurdles for larger systems. The bound on the number of invariants needed to classify fusion categories can almost certainly be improved. For $B_{p,2}$ as well as the $Rep(D(S_3))$ examples of [HT15] there is a single geometric invariant which distinguishes them. For fusion categories with additional structure, specifically modular categories, there are already conjectures about classifying invariants [BNRW13].

I also want to contribute to other approaches for constructing and classifying fusion categories. By solving the pentagon equations and computing invariants I can show there are only eight classes of six object Haagerup categories (four untwisted [**?**] and four twisted [MPS15]). However, a better way to do the proof will be by expanding [**?**, **?**] and related work, of which I want to be a part.

4 Enumerating Small Rank Fusion Categories

Concrete examples are needed to develop insight for, among other things, constructing invariants. To this end, T. Hagge and I have developed the FusionCategories [HT] package to solve for and manipulate the arithmetic data (B, N, F). This includes a database of solutions for most known categories up to rank 5. Individual examples have Frobenius-Perron dimension up to about 40 and rank up to 12.

Even for small rank systems, the pentagon equations can present interesting computer algebra challenges. An example is those coming from the rank 4 based ring $K_2(2)$ [Lar14]. $K_2(2)$ has multiplicity and sits just outside current capabilities. Rings with multiplicity are important because there are so few examples of solutions. As progress on this M. Cheng and I produced arithmetic data for the $SO(8)_1 \times \mathbb{Z}_3$ ring which has multiplicity. These were used in [BBCW14].

I have also worked on skeletalizing module categories [Ost03a], e.g. $\mathcal{M}$, over a fusion category $\mathcal{C}$ and so that their arithmetic data can be computed. As part of the FusionCategories package, we have prototypical code for explicitly constructing algebra objects in a fusion category $\mathcal{C}$ and determining their (right) modules. This can then be used to determine based modules for $\mathcal{K}_0(\mathcal{C})$. Given arithmetic data for

$\mathcal{C}$, we can solve for the (left) module structure of $\mathcal{M}$ over $\mathcal{C}$. To start with we've computed the arithmetic data for module categories over $Rep(S_3)$.

4.1 Future Work

Solving pentagon equations is a hard, important problem and there there is a lot we can learn about fusion categories by looking at their solutions. There are plenty of examples from which to work (see [GK95] among others), many involving multiplicity. A hard but probably achievable near term goal is to determine arithmetic data for all rank 4 fusion categories.

Related to this is developing new methods for constructing categories with multiplicity. Completing the classification of $SO(8)_1 \times \mathbb{Z}_3$ categories will be useful in automating choice of basis for > 1 dimensional hom spaces. Regardless though, new computer algebra methods are certainly necessary.

There is a lot to do in continuing to improve the FusionCategories package. Recently, progress has been made towards skeletalizing other structures related to fusion categories. In [BBCW14] a skeletalization for G-crossed braided categories was constructed so computing further examples of G-crossed braidings is an example. Another task is to be able to compute right module structures of $\mathcal{M}$ over $\mathcal{D}$ and subsequently whether or not $\mathcal{M}$ is a $\mathcal{C} - \mathcal{D}$ bimodule category.

5 Grothendieck Rings

To determine categorifiable based rings, one starts with B, picks a *-operation capturing the duality, and constructs diophantine equations which give based rings. One then imposes additional conditions - e.g. that formal codegrees are d-numbers [Ost08]. A complete classification of categorifiable based rings exists at rank 2 [Ost03b] and partial classifications exist for 3 - 5 [Ost13,Lar14,Bru12,RSW09,BNRW15]. With R. Johnson, S. Morrisson, S.-H. Ng, D.Penneys, J. Roat, and H. Tucker, I have worked on classifying pseudo-unitary categorifiable self dual rank 4 based rings.

5.1 Future Work

In completing rank 4 and beyond, new tools and approaches are needed. The problem is again solving polynomial equations taking values in a ring and geometric insights should be of use. As an example, one can determine the possible parities of structure coefficients by solving things over $\mathbb{Z}_2$. Alternatively, one could consider solutions over $\mathbb{C}$ so that based rings correspond to lattice points.

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